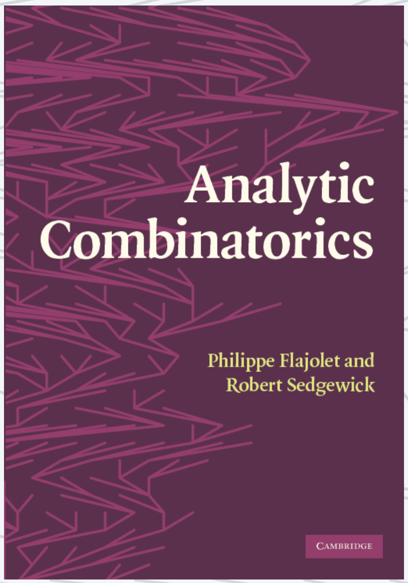
ANALYTIC COMBINATORICS

PART TWO



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7. Applications of Singularity Analysis

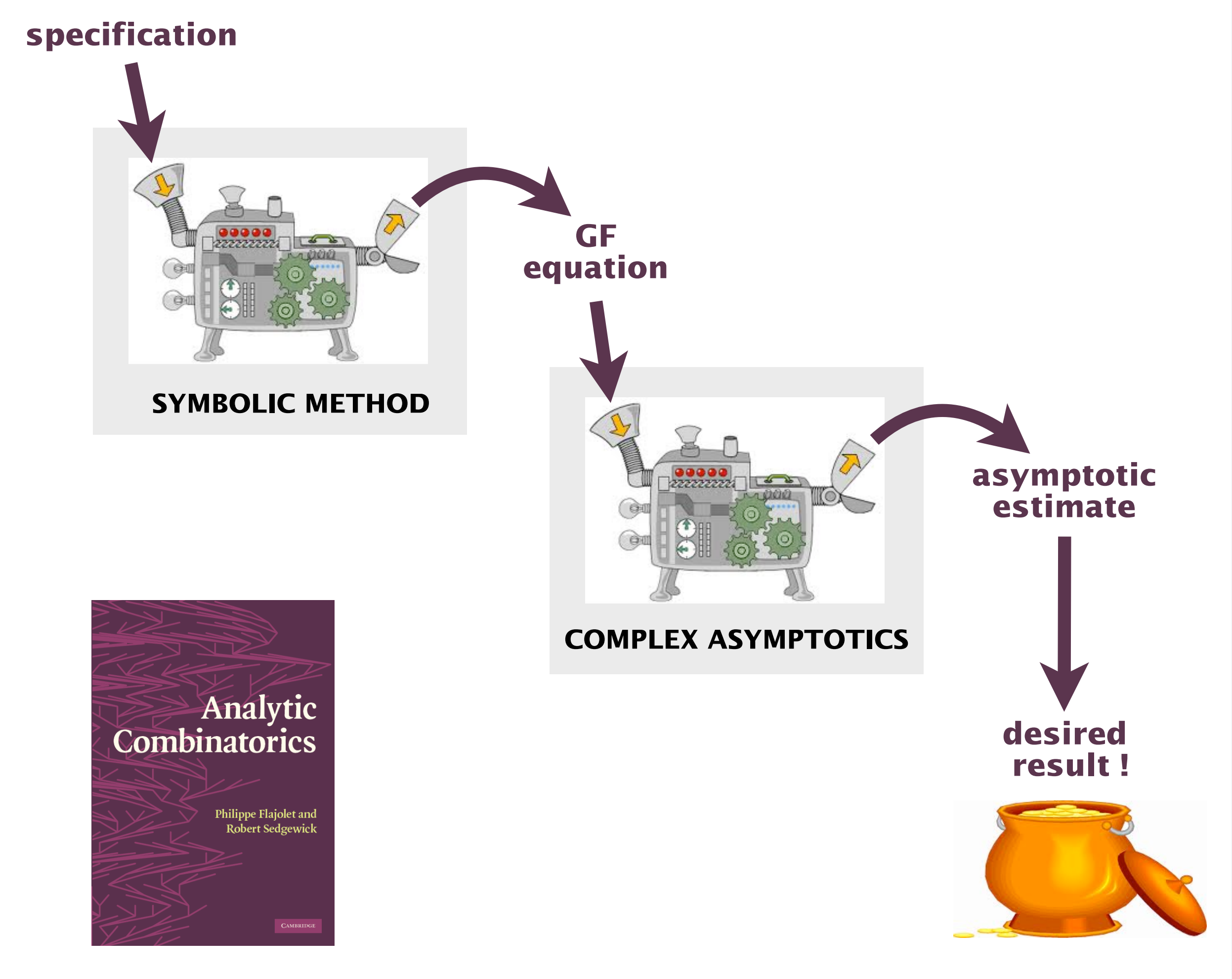
Analytic combinatorics overview

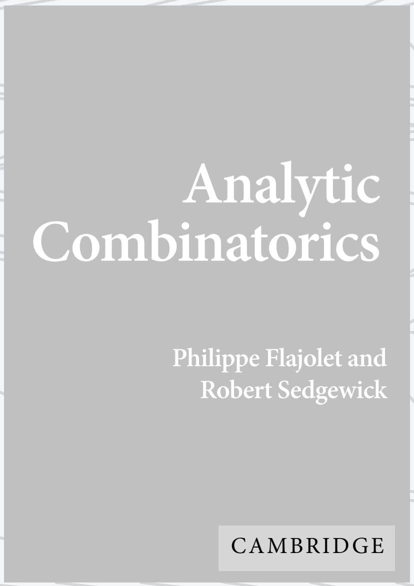
A. SYMBOLIC METHOD

1. OGFs
2. EGFs
3. MGFs

B. COMPLEX ASYMPTOTICS

4. Rational & Meromorphic
5. Applications of R&M
6. Singularity Analysis
7. Applications of SA
8. Saddle point





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7. Applications of Singularity Analysis

- **Simple varieties of trees**
- Labelled sets
- Mappings
- Tree-like classes

Transfer theorem for invertible tree classes

[from Lecture 6]

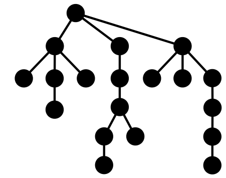
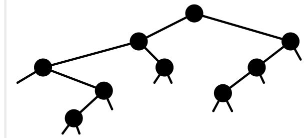
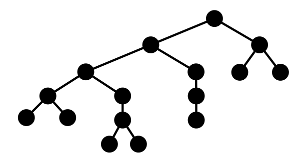
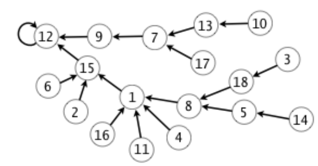
Theorem. If a simple variety of trees $\mathbf{F} = \mathbf{Z}$ [× or ★] SEQ$_\Phi$($\mathbf{F}$) is λ-invertible where the GF satisfies $F(z) = z\phi(F(z))$ and is the positive real root of $\phi(\lambda) = \lambda\phi'(\lambda)$ then

$$[z^N]F(z) \sim \frac{1}{\sqrt{2\pi\phi''(\lambda)/\phi(\lambda)}}\phi'(\lambda)^N N^{-3/2}$$

and $F(z) \sim \lambda - \sqrt{2\phi(\lambda)/\phi''(\lambda)}\sqrt{1 - z\phi'(\lambda)}$

Important note: Singularity analysis gives *both*
- Coefficient asymptotics.
- Asymptotic estimate of GF near dominant singularity.

applications

general trees	
binary trees	
unary-binary trees	
Cayley trees	

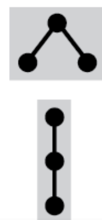
[and many, many more...]

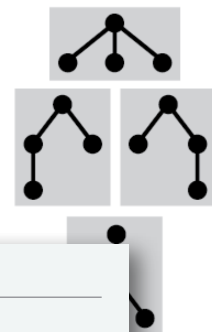
Example 1: Rooted ordered trees

Q. How many trees with N nodes?

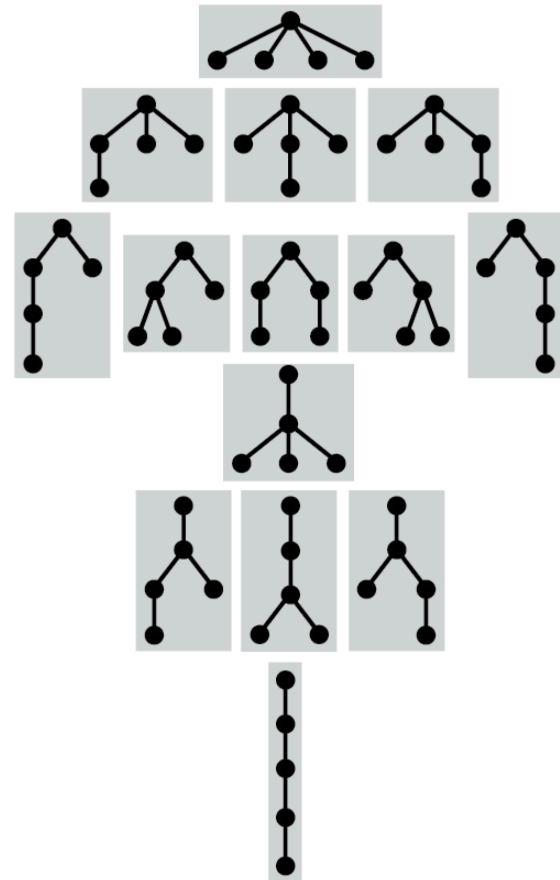
$G_1 = 1$

$G_2 = 1$





5

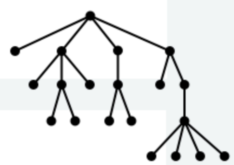


$G_5 = 14$

How many trees with N nodes?

Symbolic method

Combinatorial class	G, the class of all trees
Construction	$G = \bullet \times SEQ(G)$ ← "a tree is a node and a sequence of trees"
OGF equation	$G(z) = z(1 + G(z) + G(z)^2 + G(z)^3 + \ldots) = \frac{z}{1 - G(z)}$
	$G(z) - G(z)^2 = z$
Quadratic equation	$G(z) = \frac{1 + \sqrt{1 - 4z}}{2}$

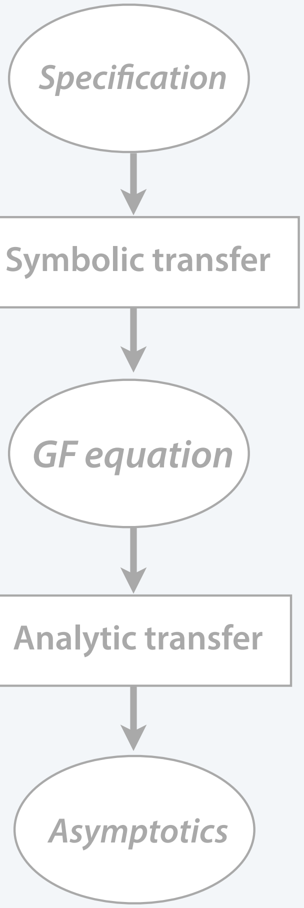


Classic next steps

Binomial theorem	$G(z) = -\frac{1}{2}\sum_{N\geq 1}\binom{\frac{1}{2}}{N}(-4z)^N$
Extract coefficients	$G_N = -\frac{1}{2}\binom{\frac{1}{2}}{N}(-4)^N = \frac{1}{N}\binom{2N-2}{N-1} = \frac{1}{4N-2}\binom{2N}{N}$
Stirling's approximation	$\sim \frac{1}{4N}\exp(2N\ln(2N) - 2N + \ln\sqrt{4\pi N} - 2(N\ln(N) - N + \ln\sqrt{2\pi N}))$
Simplify	$G_N \sim \frac{4^{N-1}}{\sqrt{\pi N^3}}$

detailed calculations omitted

Example 1: Rooted ordered trees



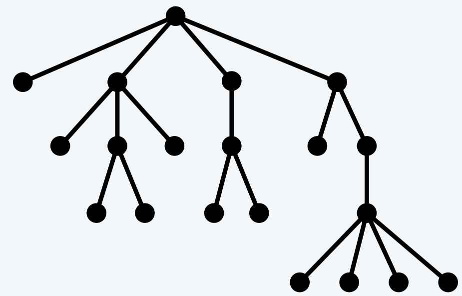
G, the class of rooted ordered trees

$$\mathbf{G} = \mathbf{Z} \times \mathrm{SEQ}(\mathbf{G})$$

$$G(z) = \frac{z}{1 - G(z)}$$

simple variety of trees

$$G_N \sim \frac{1}{4\sqrt{\pi}} 4^N N^{3/2}$$



Theorem. If a simple variety of trees $\mathbf{F} = \mathbf{Z}$ [× or ★] $\mathrm{SEQ}_\Phi(\mathbf{F})$ is λ-invertible where the GF satisfies $F(z) = z\phi(F(z))$ and is the positive real root of $\phi(\lambda) = \lambda\phi'(\lambda)$ then

$$[z^N]F(z) \sim \frac{1}{\sqrt{2\pi\phi''(\lambda)/\phi(\lambda)}} \phi'(\lambda)^N N^{-3/2}$$

$$\phi(u) = \frac{1}{1-u}$$
$$\phi'(u) = \frac{1}{(1-u)^2}$$
$$\phi''(u) = \frac{1}{(1-u)^3}$$

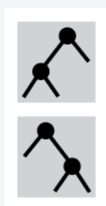
$$\frac{1}{1-\lambda} = \frac{\lambda}{(1-\lambda)^2}$$

$$\lambda = 1/2$$
$$\phi(\lambda) = 2$$
$$\phi'(\lambda) = 4$$
$$\phi''(\lambda) = 16$$

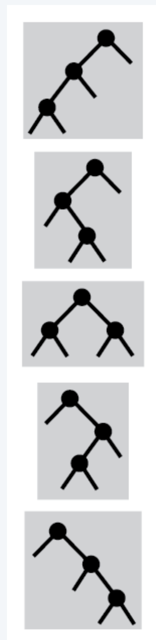
Example 2: Binary trees

How many binary trees with N nodes?

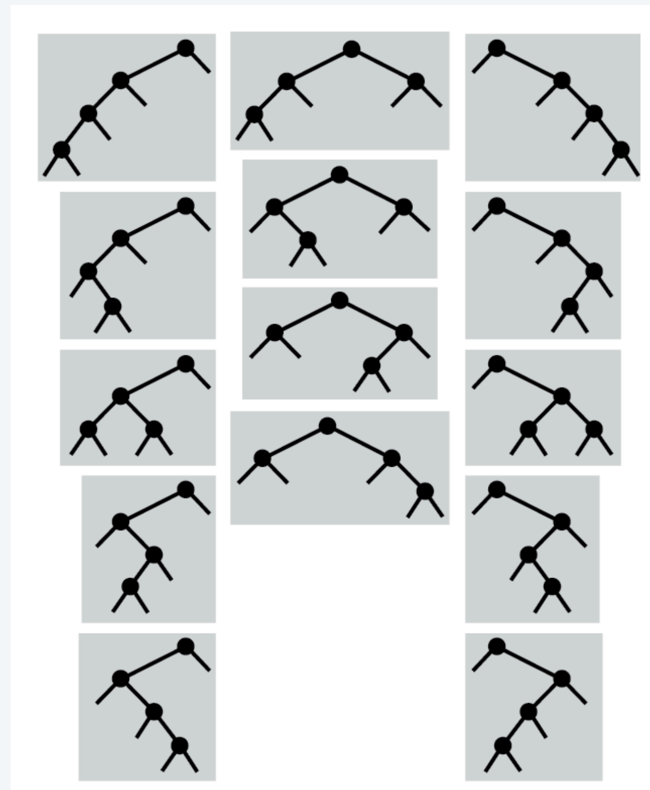
$T_1 = 1$



$T_2 = 2$

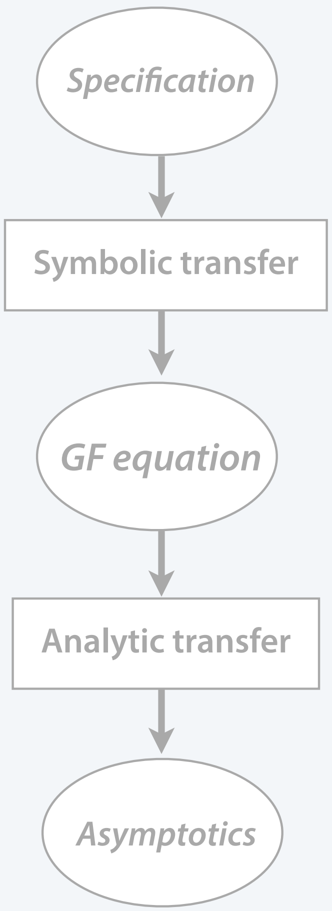


$T_3 = 5$

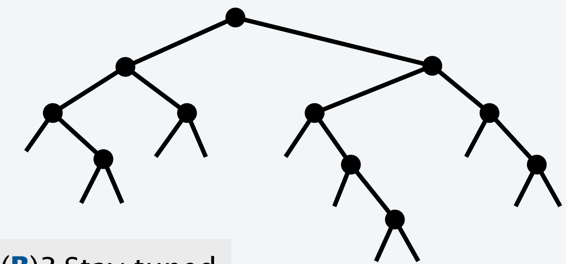


$T_4 = 14$

Example 2: Binary trees



B, the class of binary trees



B = • × (**E** + **B**) × (**E** + **B**)

Expecting **B** = • + • × $\text{SEQ}_{0,2}$(**B**)? Stay tuned.

$$B(z) = z(1 + B(z))^2$$

simple variety of trees

$$[z^N]B(z) \sim \frac{1}{\sqrt{\pi}} 4^N N^{3/2}$$

Theorem. If a simple variety of trees **F** = **Z** [× or ★] SEQ_ϕ(**F**) is λ-invertible where the GF satisfies $F(z) = z\phi(F(z))$ and is the positive real root of $\phi(\lambda) = \lambda\phi'(\lambda)$ then

$$[z^N]F(z) \sim \frac{1}{\sqrt{2\pi\phi''(\lambda)/\phi(\lambda)}} \phi'(\lambda)^N N^{-3/2}$$

$$\phi(u) = (1+u)^2$$
$$\phi'(u) = 2(1+u)$$
$$\phi''(u) = 2$$

$$(1+\lambda)^2 = 2\lambda(1+\lambda)$$

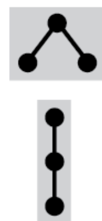
$$\lambda = 1$$
$$\phi(\lambda) = 4$$
$$\phi'(\lambda) = 4$$
$$\phi''(\lambda) = 2$$

Example 3: Unary-binary trees

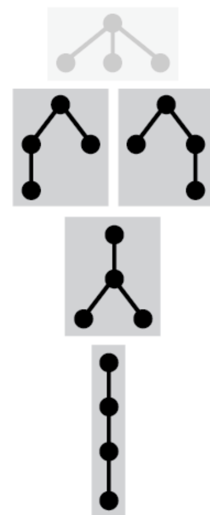
Q. How many unary-binary trees with N nodes?

$M_1 = 1$

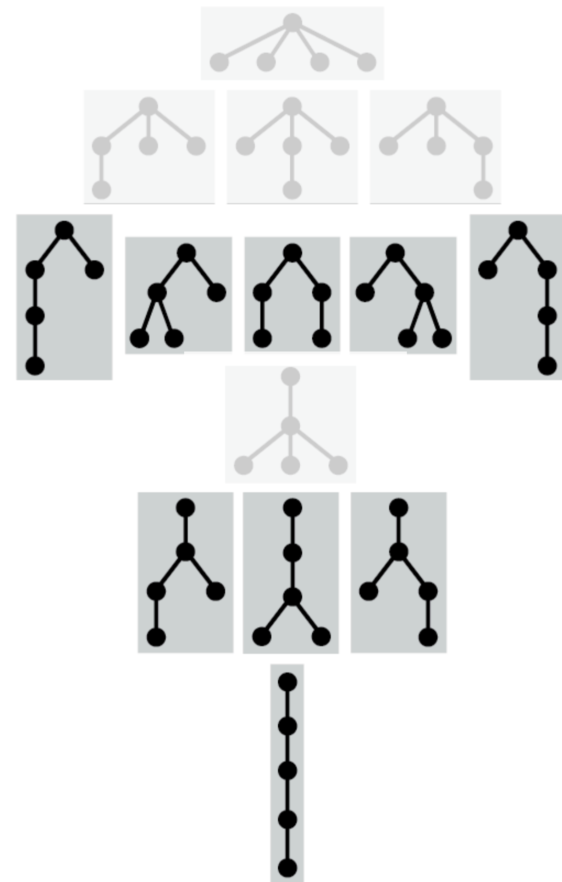
$M_2 = 1$



$M_3 = 2$



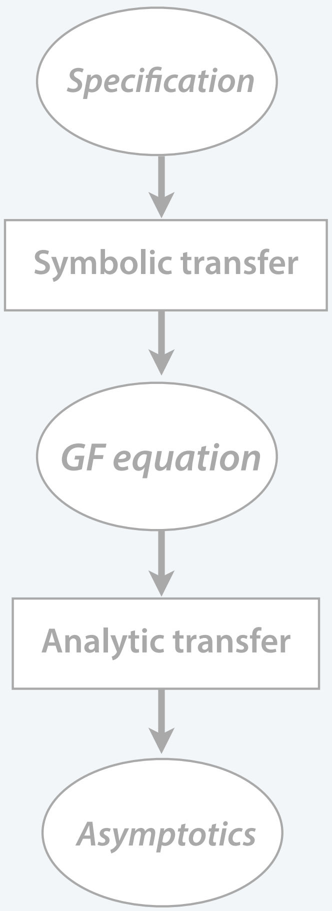
$M_4 = 4$



$M_5 = 9$

degrees of all nodes 0, 1, or 2

Example 3: Unary-binary trees



M, the class of all unary-binary trees

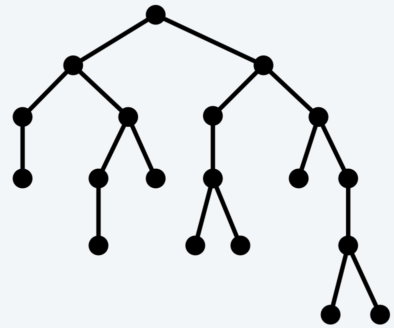
$$\mathbf{M} = \mathbf{Z} \times SEQ_{0,1,2}(\mathbf{M})$$

↓

$$M(z) = z(1 + M(z) + M(z)^2)$$

↓ *simple variety of trees*

$$M_N \sim \frac{1}{\sqrt{4\pi/3}} 3^N N^{-3/2}$$



Theorem. If a simple variety of trees $\mathbf{F} = \mathbf{Z}$ [× or ★] $\mathrm{SEQ}_\Phi(\mathbf{F})$ is λ-invertible where the GF satisfies $F(z) = z\phi(F(z))$ and is the positive real root of $\phi(\lambda) = \lambda\phi'(\lambda)$ then

$$[z^N]F(z) \sim \frac{1}{\sqrt{2\pi\phi''(\lambda)/\phi(\lambda)}} \phi'(\lambda)^N N^{-3/2}$$

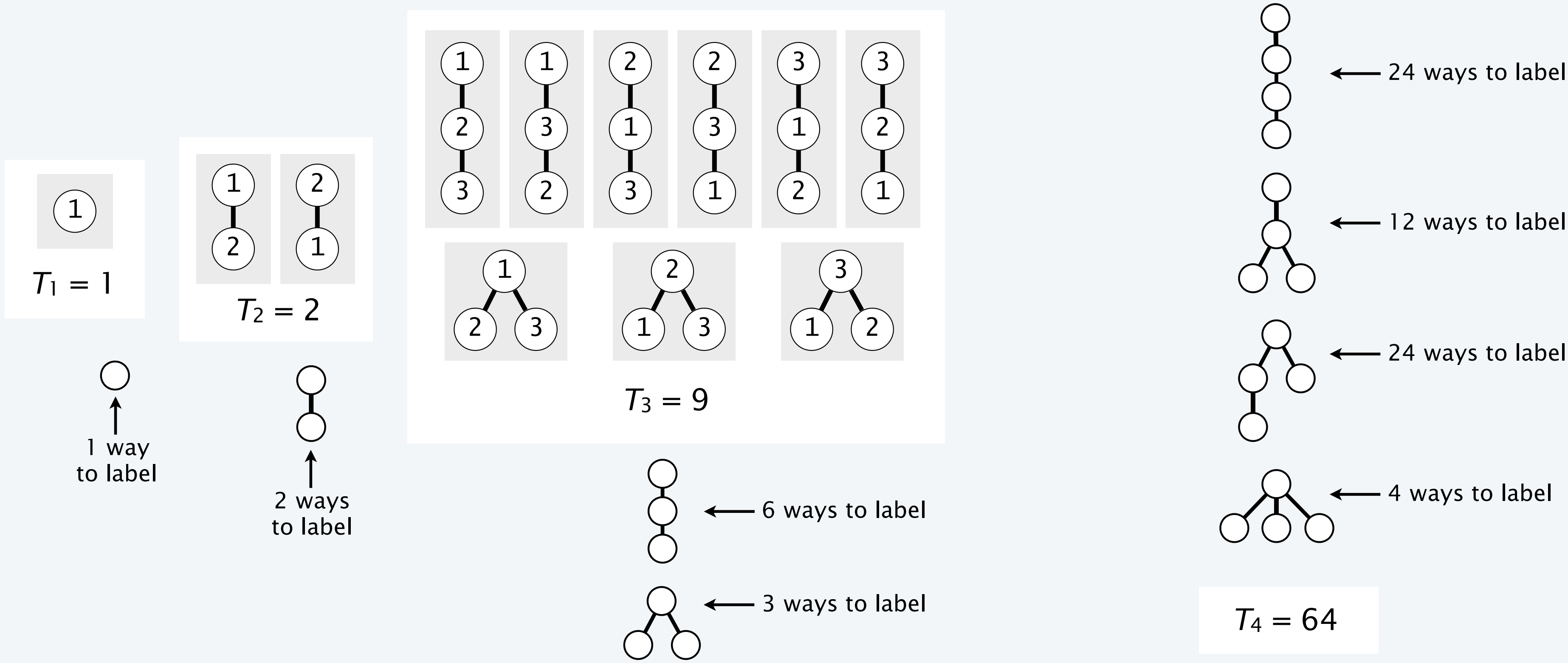
$$\phi(u) = 1 + u + u^2$$
$$\phi'(u) = 1 + 2u$$
$$\phi''(u) = 2$$

$$1 + \lambda + \lambda^2 = \lambda + 2\lambda$$

$$\lambda = 1$$
$$\phi(\lambda) = 3$$
$$\phi'(\lambda) = 3$$
$$\phi''(\lambda) = 2$$

Example 4: Cayley trees

Q. How many different labelled rooted *unordered* trees of size N?



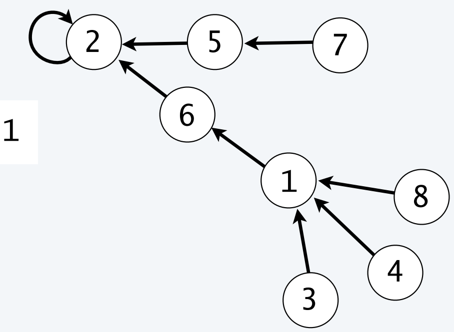
A. N^{N-1}. (See EGF lecture.)

Example 4: Cayley trees (exact, from EGF lecture)

Class	*C*, the class of labelled rooted unordered trees
EGF	$C(z) = \sum_{c \in \mathcal{C}} \frac{z^{\|c\|}}{\|c\|!} \equiv \sum_{N \geq 0} C_N \frac{z^N}{N!}$

Example

6 2 1 1 2 2 5 1



Construction $\quad C = Z \star (SET(C))$ ← "a tree is a root connected to a set of trees"

EGF equation $\quad C(z) = ze^{C(z)}$

Extract coefficients by Lagrange inversion with $f(u) = u/e^u$

$$[z^N]C(z) = \frac{1}{N}[u^{N-1}]\Big(\frac{u}{u/e^u}\Big)^N$$

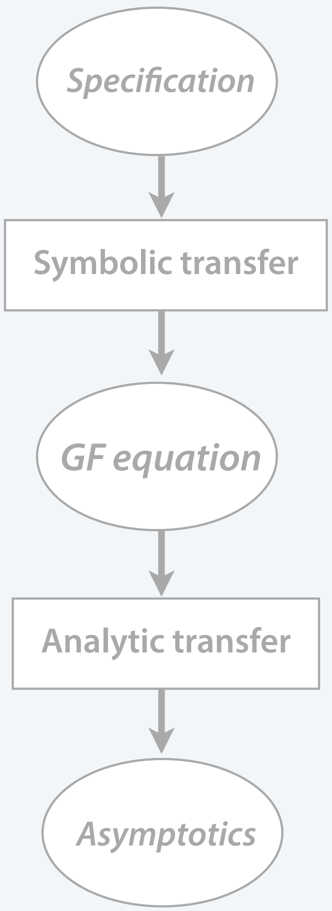
$$= \frac{1}{N}[u^{N-1}]e^{uN} = \frac{N^{N-1}}{N!}$$

$$C_N = N![z^N]C(z) = N^{N-1}$$ ✓

Lagrange Inversion Theorem.

If a GF $g(z) = \sum_{n \geq 1} g_n z^n$ satisfies the equation $z = f(g(z))$ with $f(0) = 0$ and $f'(0) \neq 0$ then $g_n = \frac{1}{n}[u^{n-1}]\Big(\frac{u}{f(u)}\Big)^n$.

Example 4: Cayley trees



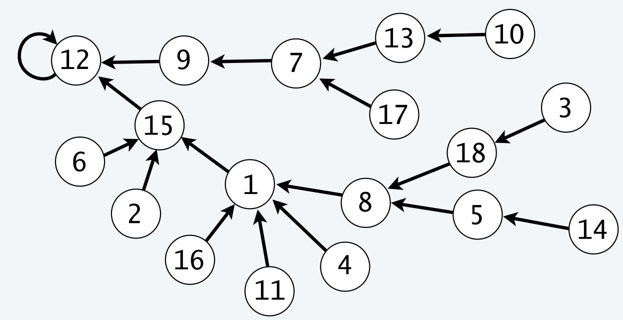
C, the class of all labelled rooted unordered trees

C = **Z** ★ *SET* (**C**)

$$C(z) = ze^{C(z)}$$

simple variety of trees

$$[z^N]C(z) = \frac{1}{\sqrt{2\pi}} e^N N^{-3/2}$$



Theorem. If a simple variety of trees **F** = **Z** [× or ★] SEQ_Φ(**F**) is λ-invertible where the GF satisfies $F(z) = z\phi(F(z))$ and is the positive real root of $\phi(\lambda) = \lambda\phi'(\lambda)$ then

$$[z^N]F(z) \sim \frac{1}{\sqrt{2\pi\phi''(\lambda)/\phi(\lambda)}} \phi'(\lambda)^N N^{-3/2}$$

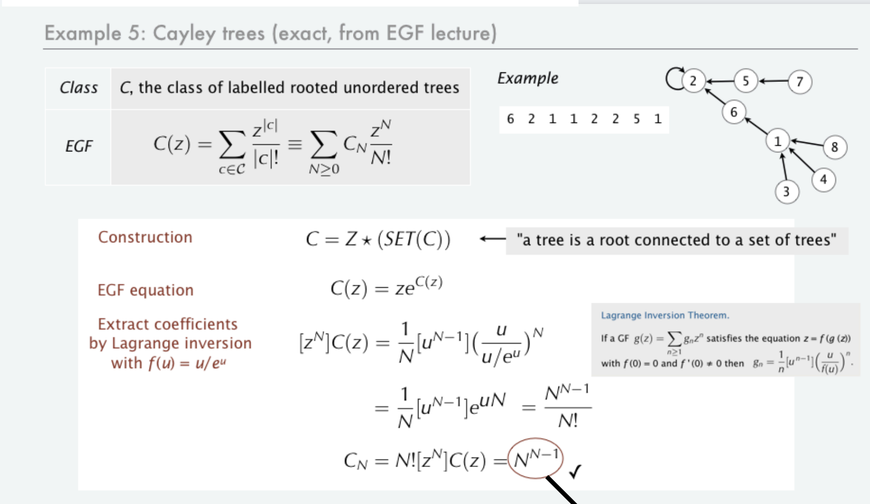
$$\phi(u) = e^u$$
$$\phi'(u) = e^u$$
$$\phi''(u) = e^u$$

$$e^\lambda = \lambda e^\lambda$$

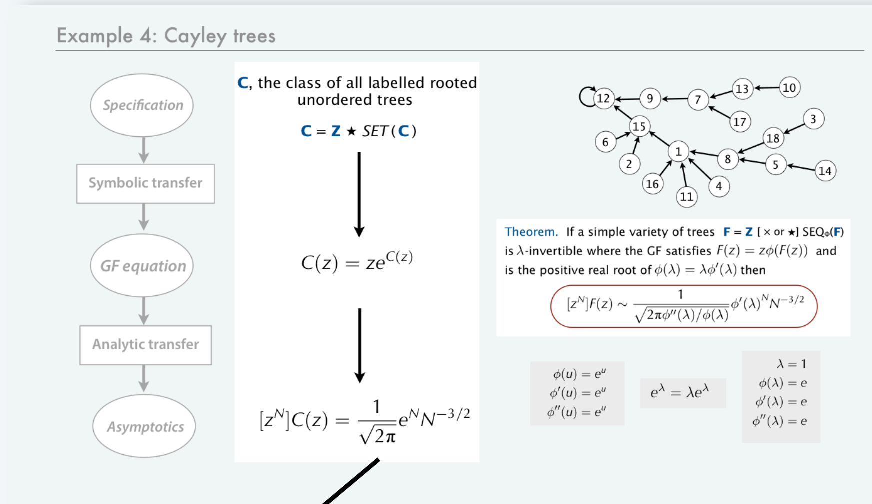
$$\lambda = 1$$
$$\phi(\lambda) = e$$
$$\phi'(\lambda) = e$$
$$\phi''(\lambda) = e$$

Aside: Stirling's formula via Cayley tree enumeration

Exact, via Lagrange inversion

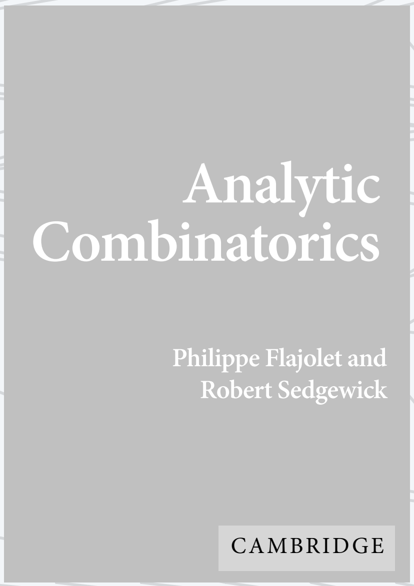


Approximate, via singularity analysis



$$N^{N-1} \sim N! \frac{e^N}{\sqrt{2\pi N^3}}$$

Theorem. $N! \sim \sqrt{2\pi N}\left(\frac{N}{e}\right)^N$ ← Stirling's formula



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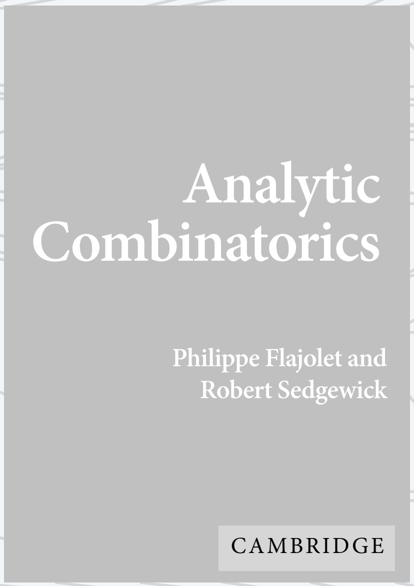
7. Applications of Singularity Analysis

- **Simple varieties of trees**
- Labelled sets
- Mappings
- Tree-like classes



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7. Applications of Singularity Analysis

- Simple varieties of trees
- **Labelled sets**
- Mappings
- Tree-like classes

Transfer theorem for exp-log labelled set classes

[from Lecture 6]

Theorem. *Asymptotics of exp-log labelled sets.*

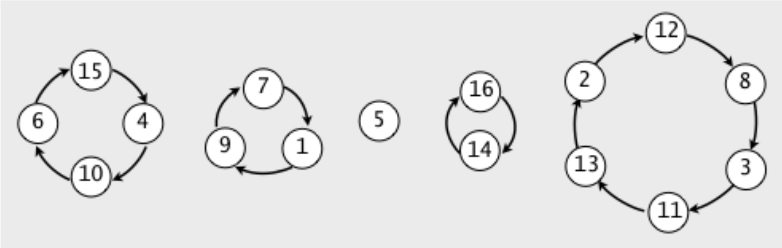
Suppose that a labelled set class **F** = SET$_\Phi$(**G**) is exp-log(α, β, ρ)

with $G(z) \sim \alpha \log \dfrac{1}{1 - z/\rho} + \beta$. Then $F(z) \sim e^{\beta}\Big(\dfrac{1}{1 - z/\rho}\Big)^{\alpha}$

and $$[z^N]F(z) \sim \frac{e^{\beta}}{\Gamma(\alpha)}\Big(\frac{1}{\rho}\Big)^{N} N^{1-\alpha}$$

Corollary. The expected number of *G*-components in a random *F*-object of size *N* is ~α ln *N.*

and is concentrated there



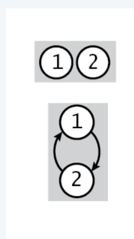
Example 5: Cycles in permutations

Q. How many permutations of *N* elements?

Q. How many cycles in a random permutation of *N* elements?

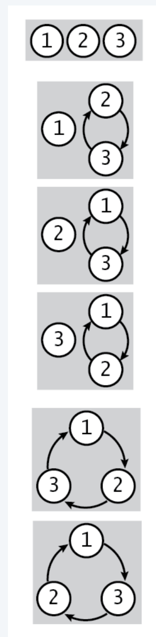
$P_1 = 1$

avg. # cycles: **1**



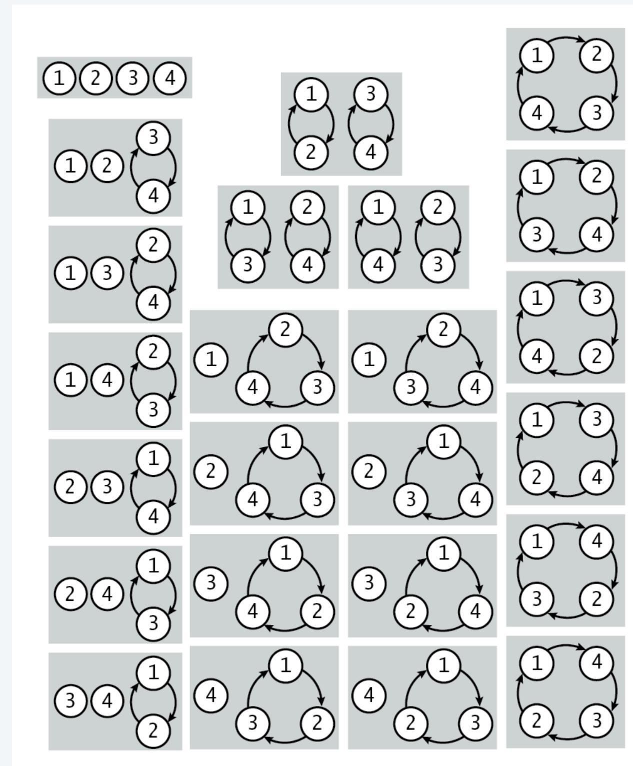
$P_2 = 2$

avg. # cycles: **1.5**



$P_3 = 6$

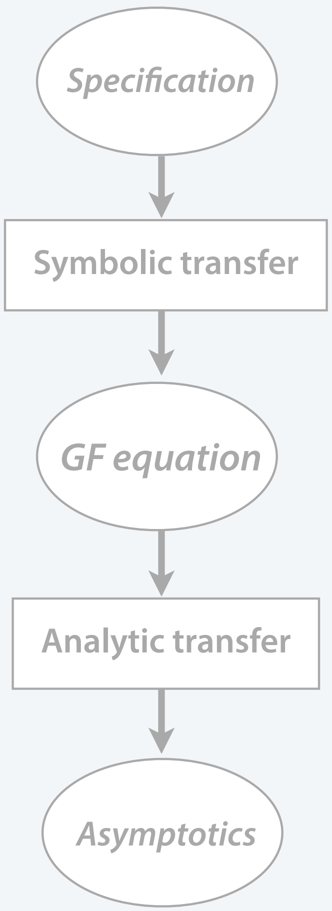
avg. # cycles: **1.8333**



$P_4 = 24$

avg. # cycles: **2.08333**

Example 5: Cycles in permutations



P, the class of all permutations

$$\mathbf{P} = \mathrm{SET}(\mathrm{CYC}(\mathbf{Z}))$$

↓

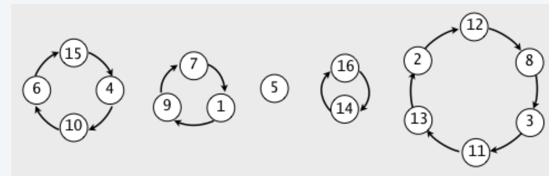
$$P(z) = \exp(\ln \frac{1}{1-z})$$

↓ *exp-log*

$$[z^N]P(z) \sim 1$$

\# permutations: $\sim N!$

avg # cycles: $\sim \ln N$



Theorem. *Asymptotics of exp-log labelled sets.*

Suppose that a labelled set class **F** = SET_Φ(**G**) is exp-log(α, β, ρ) with $G(z) \sim \alpha \log \frac{1}{1 - z/\rho} + \beta$. Then $F(z) \sim e^{\beta}\left(\frac{1}{1-z/\rho}\right)^{\alpha}$ and $[z^N]F(z) \sim \frac{e^{\beta}}{\Gamma(\alpha)}\left(\frac{1}{\rho}\right)^{N} N^{1-\alpha}$

$$\ln \frac{1}{1-z} = \alpha \log \frac{1}{1-z/\rho} + \beta$$

for $\alpha = 1, \beta = 0,$ and $\rho = 1$

Corollary. The expected number of *G*-components in a random *F*-object of size *N* is ~α ln *N*.

↑ and is concentrated there

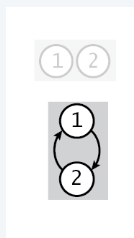
Example 6: Cycles in derangements

Q. How many derangements of N elements?

Q. How many cycles in a random derangement of N elements?

$D_1 = 0$

avg. # cycles: **0**

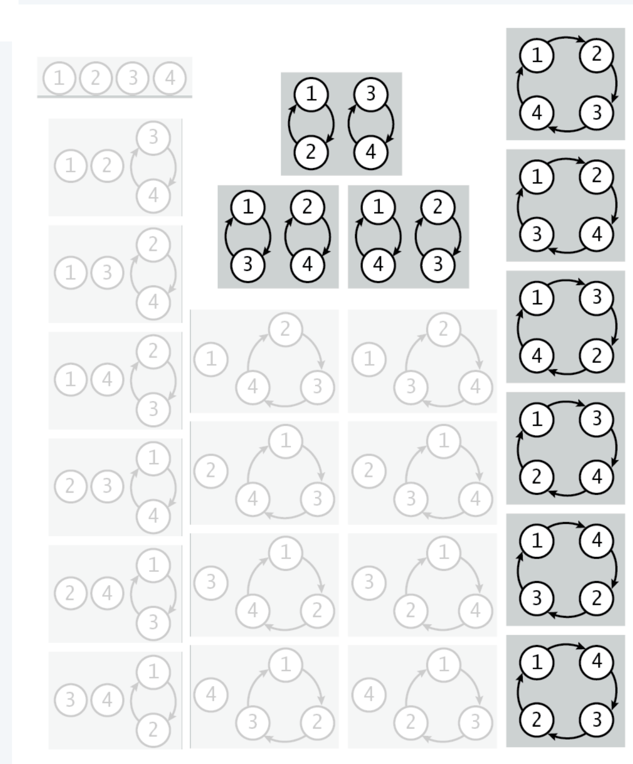


$D_2 = 1$

avg. # cycles: **1**

$D_3 = 2$

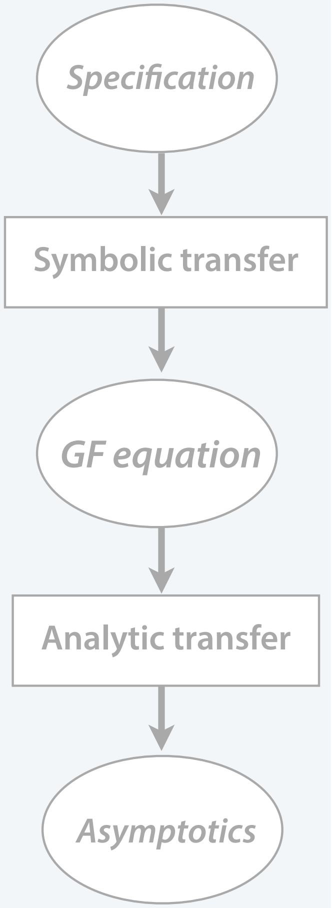
avg. # cycles: **1**



$D_4 = 9$

avg. # cycles: **1.33333**

Example 6: Cycles in derangements



D, the class of all derangements

$$\mathbf{D} = \mathrm{SET}(\mathrm{CYC}_{>0}(\mathbf{Z}))$$

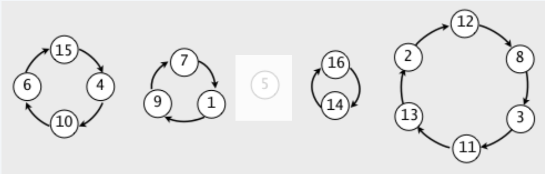
$$D(z) = \exp\left(\ln\frac{1}{1-z} - 1\right)$$

exp-log

$$[z^N]D(z) \sim e^{-1}$$

derangements: $\sim N!/e$

avg # cycles: $\sim \ln N$



Theorem. *Asymptotics of exp-log labelled sets.*

Suppose that a labelled set class $\mathbf{F} = \mathrm{SET}_\Phi(\mathbf{G})$ is exp-log(α, β, ρ) with $G(z) \sim \alpha \log \frac{1}{1-z/\rho} + \beta$. Then $F(z) \sim e^{\beta}\left(\frac{1}{1-z/\rho}\right)^{\alpha}$ and $[z^N]F(z) \sim \frac{e^{\beta}}{\Gamma(\alpha)}\left(\frac{1}{\rho}\right)^{N} N^{1-\alpha}$

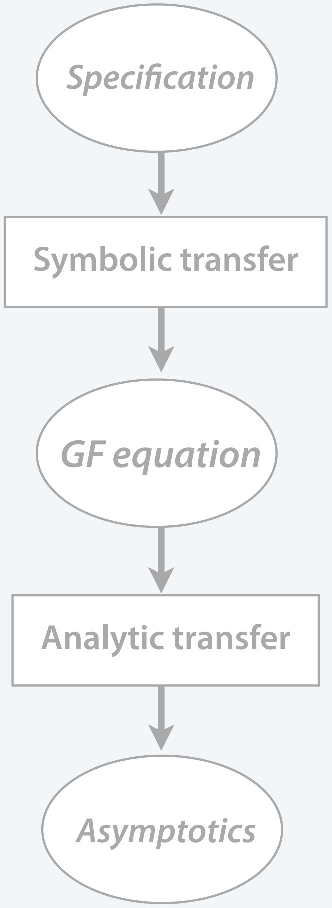
$$\ln\frac{1}{1-z} - 1 = \alpha\log\frac{1}{1-z/\rho} + \beta$$

for $\alpha = 1, \beta = -1,$ and $\rho = 1$

Corollary. The expected number of *G*-components in a random *F*-object of size *N* is ~α ln *N*.

and is concentrated there

Example 6: Cycles in generalized derangements



D, the class of all permutations
having no cycles of length w_1, w_2, ... w_t

D = SET(CYC$_{\neq wi}$(**Z**))

$$D(z) = \exp(\ln \frac{1}{1-z} - \frac{z^{w_1}}{w_1} - \ldots - \frac{z^{w_t}}{w_t})$$

exp-log

$$[z^N]D(z) = \exp(-\frac{1}{w_1} - \ldots - \frac{1}{w_t})$$

$$\ln \frac{1}{1-z} - 1 = \alpha \log \frac{1}{1-z/\rho} + \beta$$

for $\alpha = 1, \beta = -\frac{1}{w_1} - \ldots - \frac{1}{w_t}$

and $\rho = 1$

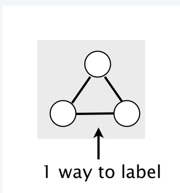
derangements: $\sim N!/e^{1/w_1 + \ldots + 1/w_t}$

avg # cycles: $\sim \ln N$

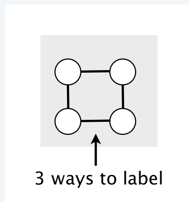
Example 7: 2-regular graphs

Q. How many labelled 2-regular graphs of N elements? ← undirected graphs with all nodes degree 2

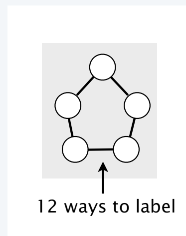
Q. How many *components* in a random 2-regular graph of N elements?



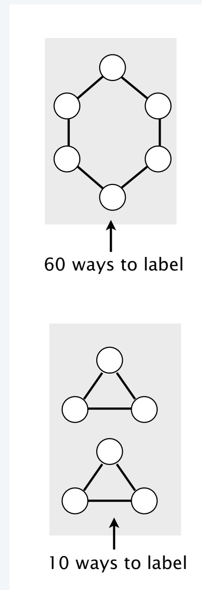
$R_3 = 1$



$R_4 = 3$

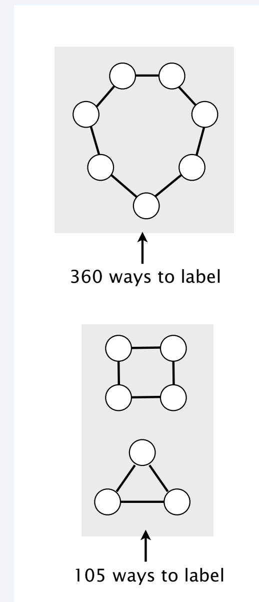


$R_5 = 12$



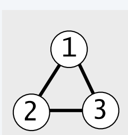
$R_6 = 70$

avg. # components:
$(1 \cdot 60 + 2 \cdot 10)/70 \doteq$ **1.143**

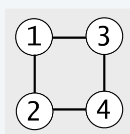


$R_7 = 465$

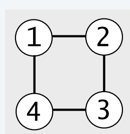
avg. # components:
$(1 \cdot 360 + 2 \cdot 105)/465 \doteq$ **1.226**



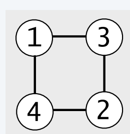
1–2
1–3
2–3



1–2
1–3
2–4
3–4

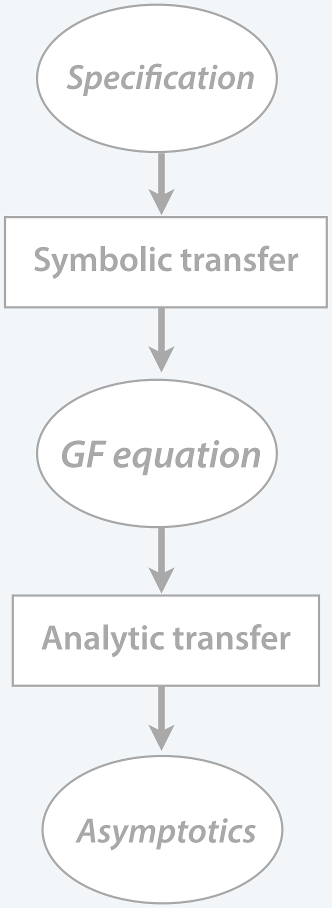


1–2
1–4
2–3
3–4



1–3
1–4
2–3
2–4

Example 7: 2-regular graphs



R, the class of 2-regular graphs

R = SET(UCYC $_{>2}$(**Z**))

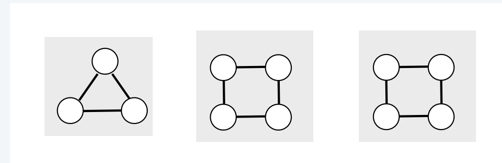
$$R(z) = \exp\left(\frac{1}{2}\ln\frac{1}{1-z} - \frac{z}{2} - \frac{z^2}{4}\right)$$

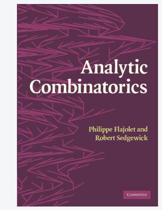
exp-log

$$[z^N]R(z) \sim \frac{e^{-3/4}}{\sqrt{\pi N}}$$

\# 2-regular graphs: $\sim N!\dfrac{e^{-3/4}}{\sqrt{\pi N}}$

avg # components: $\sim \dfrac{1}{2}\ln N$





page 133
page 449

Theorem. *Asymptotics of exp-log labelled sets.*

Suppose that a labelled set class **F** = SET$_\Phi$(**G**) is exp-log(α, β, ρ) with $G(z) \sim \alpha \log\dfrac{1}{1-z/\rho} + \beta$. Then $F(z) \sim e^{\beta}\left(\dfrac{1}{1-z/\rho}\right)^{\alpha}$

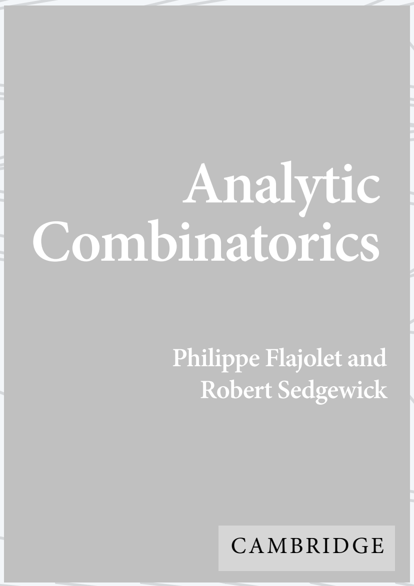
and $[z^N]F(z) \sim \dfrac{e^{\beta}}{\Gamma(\alpha)}\left(\dfrac{1}{\rho}\right)^{N} N^{1-\alpha}$

$$G(z) \sim \alpha \log\frac{1}{1-z/\rho} + \beta$$

for $\alpha = 1/2, \beta = 3/4,$ and $\rho = 1$

Corollary. The expected number of *G*-components in a random *F*-object of size *N* is ~α ln *N*.

and is concentrated there



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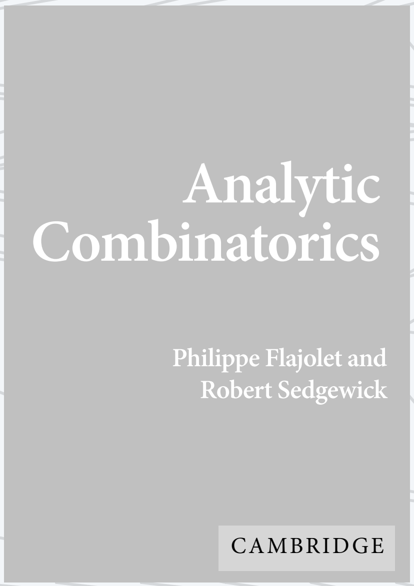
7. Applications of Singularity Analysis

- Simple varieties of trees
- **Labelled sets**
- Mappings
- Tree-like classes



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7. Applications of Singularity Analysis

- Simple varieties of trees
- Labelled sets
- **Mappings**
- Tree-like classes

Example 7: Mappings

[from Lecture 2]

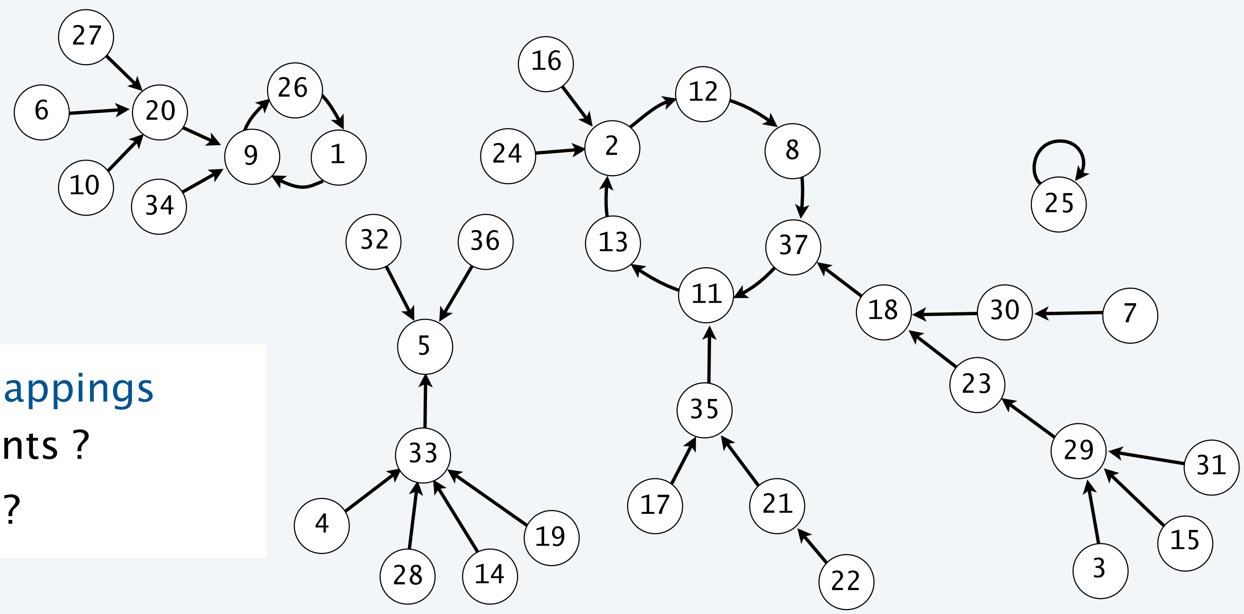
Def. A *mapping* is a function from the set of integers from 1 to N onto itself.

Example

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37
9	12	29	33	5	20	30	37	26	20	13	8	2	33	29	2	35	37	33	9	35	21	18	2	25	1	20	33	23	18	29	5	5	9	11	5	11

Every mapping corresponds to a digraph

- N vertices, N edges
- Outdegrees: all 1
- Indegrees: between 0 and N



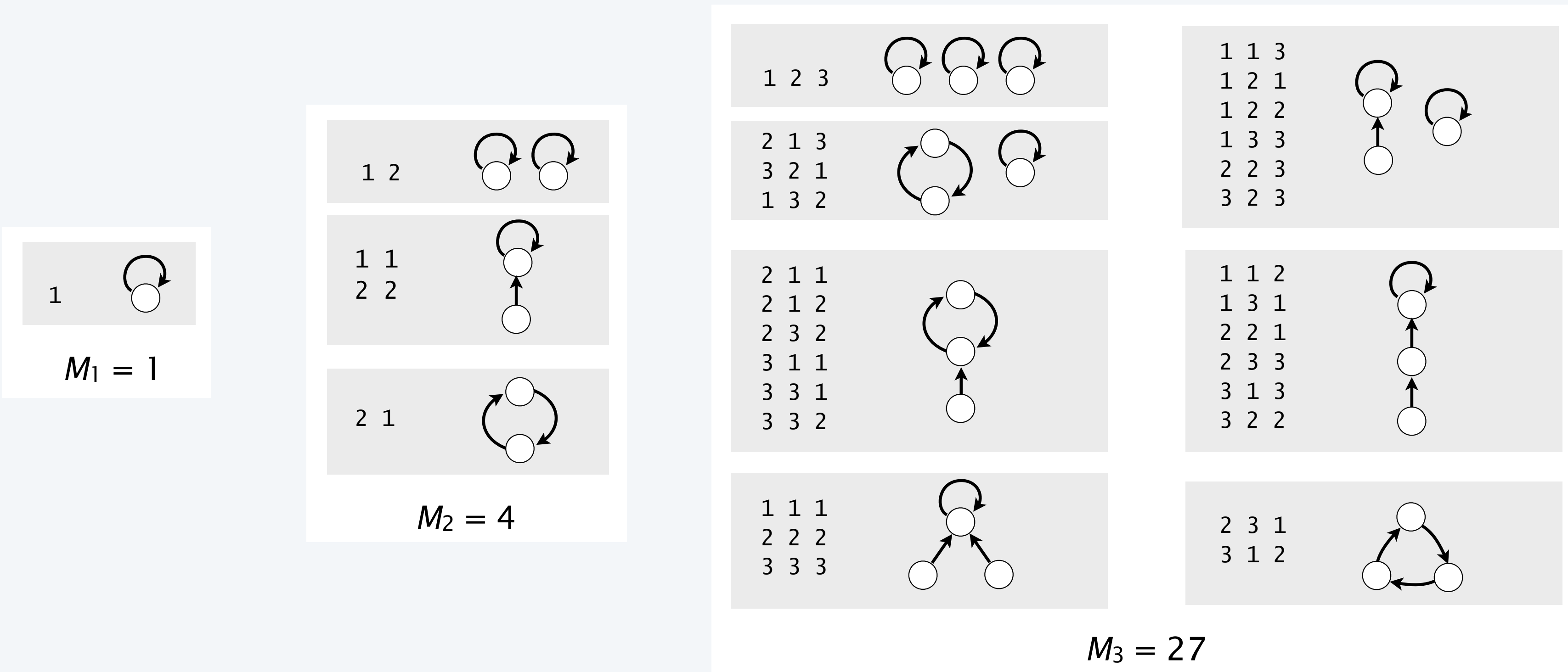
Natural questions about random mappings

- How many connected components ?
- How many nodes are on cycles ?

Mappings

[from Lecture 2]

Q. How many *mappings* of length *N* ?



A. N^N, by correspondence with *N*-words, but *internal structure is of interest*.

Mapping EGFs

[from Lecture 2]

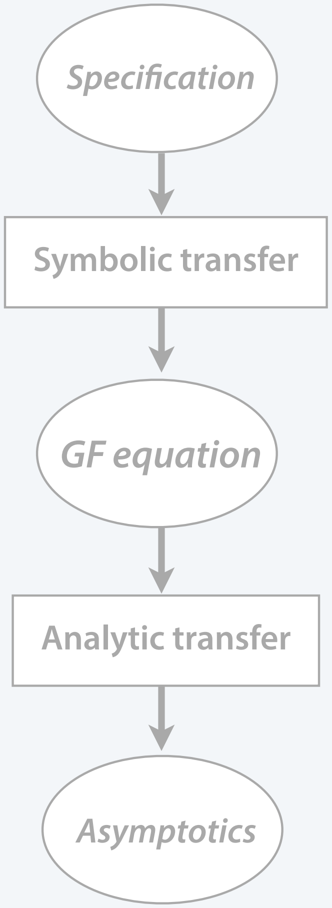
Combinatorial class	*C*, the class of Cayley trees	⟵ labelled, rooted, unordered
Construction	$C = Z \star (SET(C))$	⟵ "a tree is a root connected to a set of trees"
EGF equation	$C(z) = ze^{C(z)}$	

Combinatorial class	*Y*, the class of mapping components	
Construction	$Y = CYC(C)$	⟵ "a mapping component is a cycle of trees"
EGF equation	$Y(z) = \ln \frac{1}{1 - C(z)}$	

Combinatorial class	*M*, the class of mappings	
Construction	$M = SET(CYC(C))$	⟵ "a mapping is a set of components"
EGF equation	$M(z) = \exp\left(\ln \frac{1}{1 - C(z)}\right) = \frac{1}{1 - C(z)}$	

Example 4: Cayley trees

[from earlier in this lecture]



C, the class of all labelled rooted unordered trees

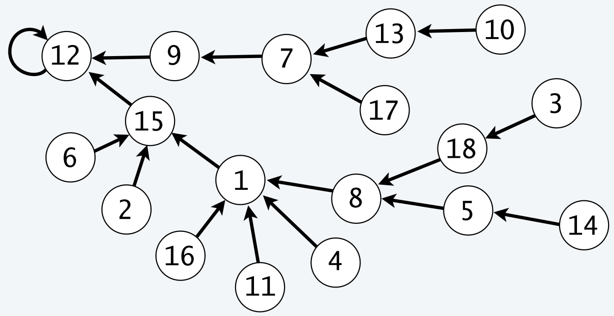
$$\mathbf{C} = \mathbf{Z} \star SET(\mathbf{C})$$

$$C(z) = ze^{C(z)}$$

$$C(z) \sim 1 - \sqrt{2}\sqrt{1 - ez}$$

simple variety of trees

$$[z^N]C(z) = \frac{1}{\sqrt{2\pi}} e^N N^{-3/2}$$



Theorem. If a simple variety of trees **F** = **Z** [× or ★] SEQ$_\phi$(**F**) is λ-invertible where the GF satisfies $F(z) = z\phi(F(z))$ and is the positive real root of $\phi(\lambda) = \lambda\phi'(\lambda)$ then

$$[z^N]F(z) \sim \frac{1}{\sqrt{2\pi\phi''(\lambda)/\phi(\lambda)}}\phi'(\lambda)^N N^{-3/2}$$

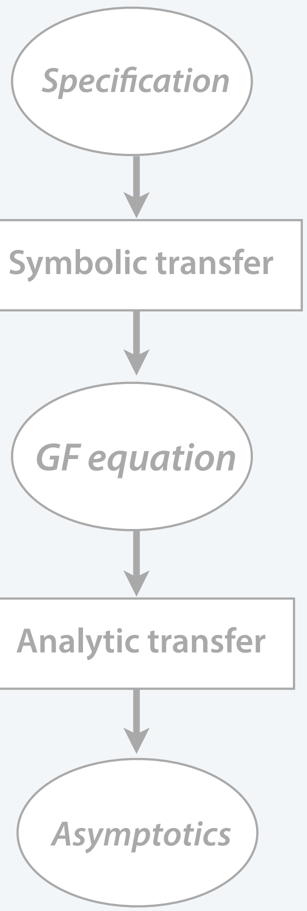
and $F(z) \sim \lambda - \sqrt{2\phi(\lambda)/\phi''(\lambda)}\sqrt{1 - z\phi'(\lambda)}$

$$\phi(u) = e^u \quad \phi'(u) = e^u \quad \phi''(u) = e^u$$

$$e^\lambda = \lambda e^\lambda$$

$$\lambda = 1 \quad \phi(\lambda) = e \quad \phi'(\lambda) = e \quad \phi''(\lambda) = e$$

Cycles of Cayley trees



Y, the class of cycles of trees (mapping components)

Y = *CYC*(**C**)

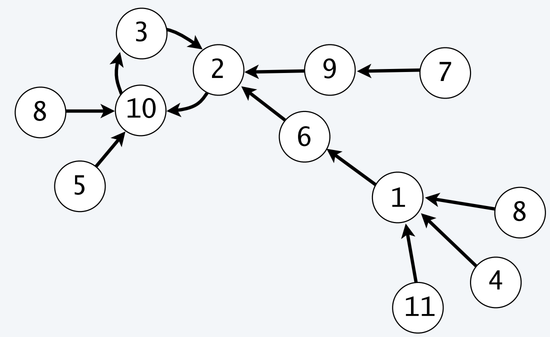
$$Y(z) = \ln\frac{1}{1 - C(z)}$$

$$\sim \frac{1}{2}\ln\frac{1}{1 - ez} - \ln\sqrt{2}$$

standard scale

$$[z^N]Y(z) \sim \frac{e^N}{2N}$$

\# cycles of trees: $\sim N!\frac{e^N}{2N} \sim \sqrt{\frac{\pi}{2N}}N^N$



1	10	2	1	10	2	9	1	2	3	1

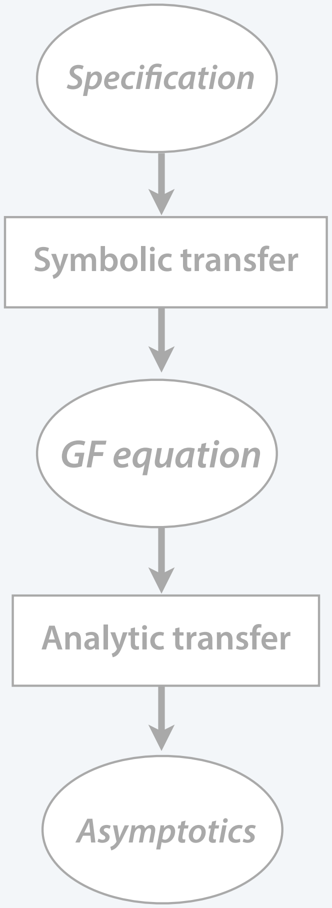
from previous slide

$$C(z) \sim 1 - \sqrt{2}\sqrt{1 - ez}$$

Stirling

$$N! \sim \sqrt{2\pi N}\left(\frac{N}{e}\right)^N$$

Mappings



M, the class of all mappings

$$\mathbf{M} = SET(\mathbf{Y})$$

↓

$$M(z) = e^{Y(z)}$$

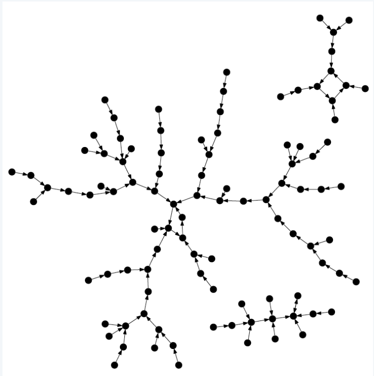
$$Y(z) \sim \frac{1}{2}\ln\frac{1}{1-ez} - \ln\sqrt{2}$$

from previous slide

↓ *exp-log*

$$N![z^N]M(z) \sim N!\frac{e^N}{\sqrt{2\pi N}}$$

$$\sim N^N \checkmark$$



Theorem. *Asymptotics of exp-log labelled sets.*

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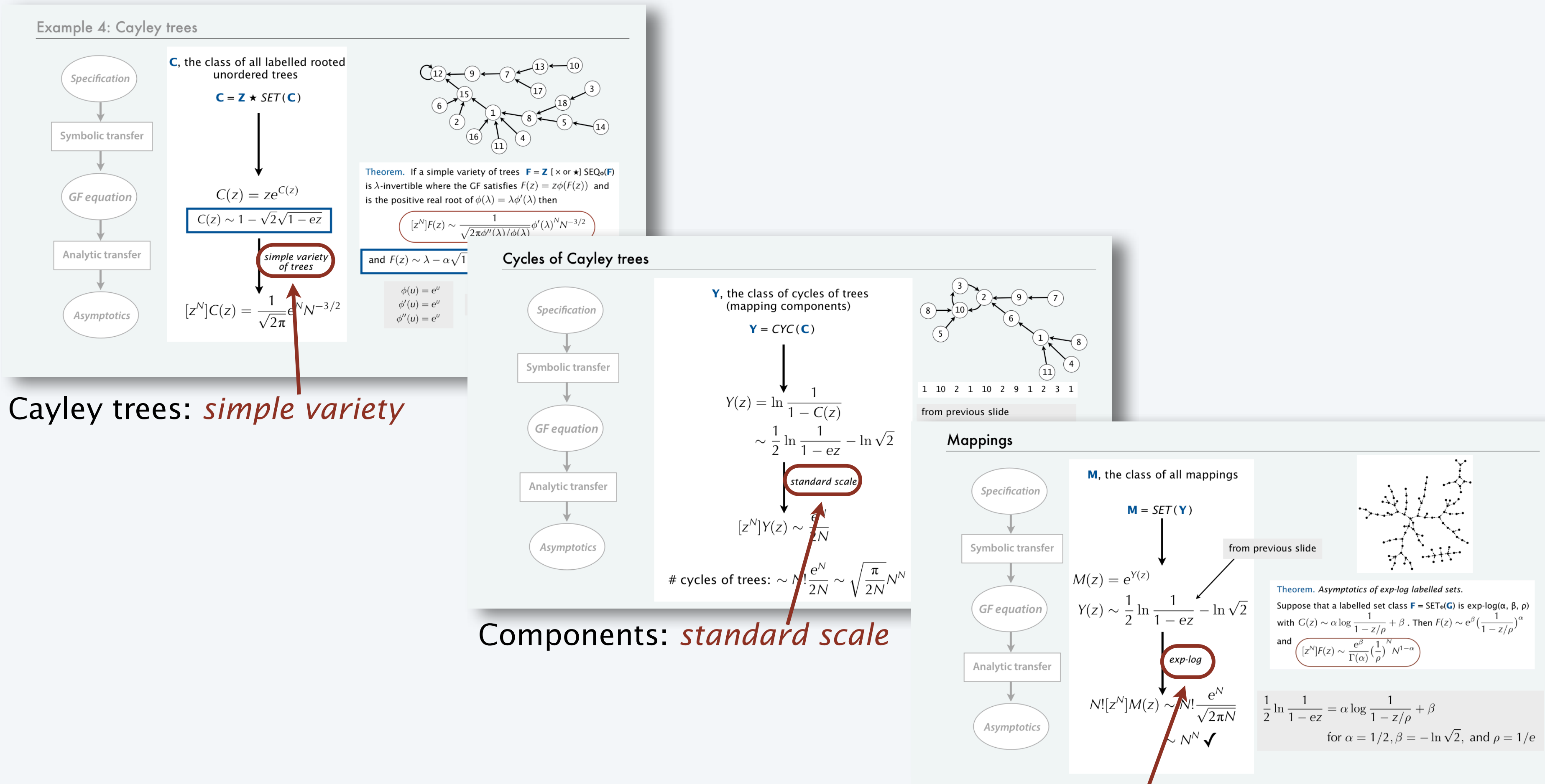
$$\frac{1}{2}\ln\frac{1}{1-ez} = \alpha\log\frac{1}{1-z/\rho} + \beta$$

for $\alpha = 1/2, \beta = -\ln\sqrt{2},$ and $\rho = 1/e$

Mappings overview



Cayley trees: *simple variety*

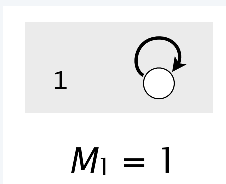
Components: *standard scale*

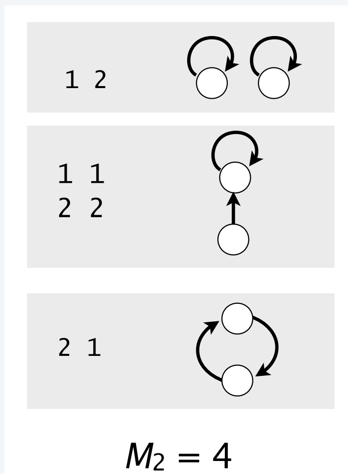
Mappings: *exp-log*

Mapping parameters

Q. How many *components* in a random mapping of length N?

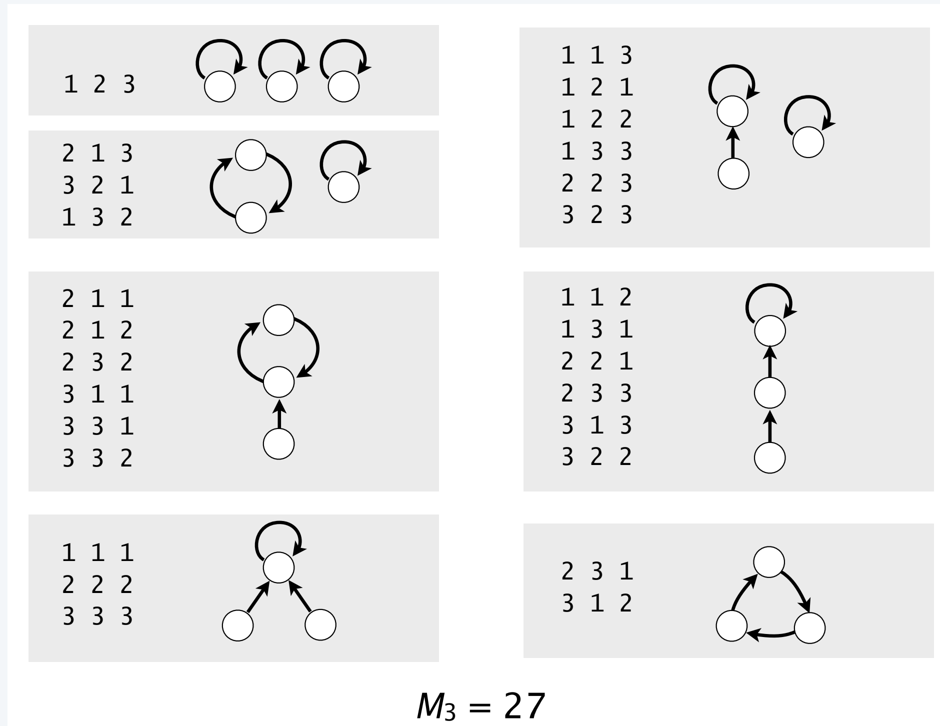
Q. How many *nodes on cycles* in a random mapping of length N?





avg. # components: **1.25**

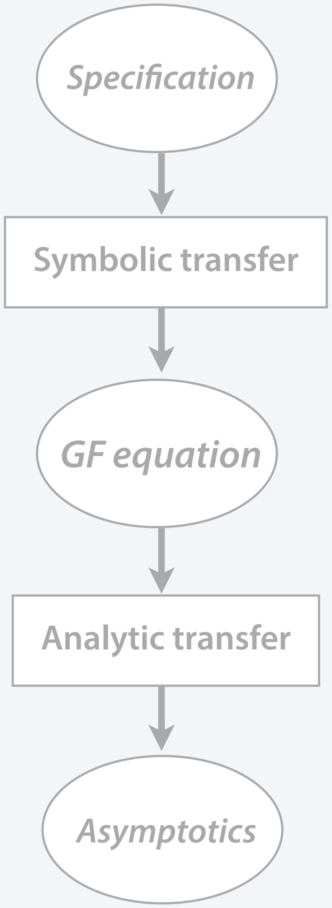
avg. # nodes on cycles: **1.5**



avg. # components: 38/27 ≐ **1.407**

avg. # nodes on cycles: 51/27 ≐ **1.889**

Components in mappings



M, the class of all mappings

M = *SET* (**Y**)

$$M(z) = e^{Y(z)}$$

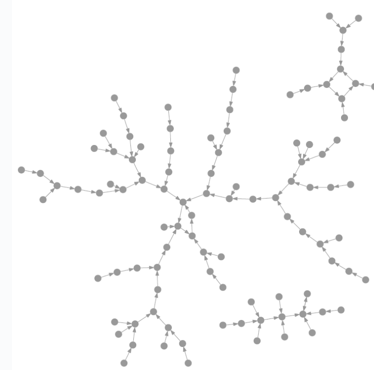
$$Y(z) \sim \frac{1}{2}\ln\frac{1}{1-ez} - \ln\sqrt{2}$$

exp-log

$$N![z^N]M(z) \sim N!\frac{e^N}{\sqrt{2\pi N}}$$

$$\sim N^N \checkmark$$

avg # components: $\frac{1}{2}\ln N$



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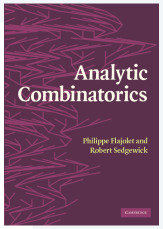
$$\frac{1}{2}\ln\frac{1}{1-ez} = \alpha\log\frac{1}{1-z/\rho} + \beta$$

for $\alpha = 1/2, \beta = -\ln\sqrt{2},$ and $\rho = 1/e$

Corollary. The expected number of *G*-components in a random *F*-object of size *N* is ~α ln *N*.

and is concentrated there

Nodes on cycles in mappings



page 462

Combinatorial class — **M**, the class of mappings

Parameter — the number of nodes on cycles (tree roots)

Construction — **M** = *SET* (*CYC* (*u* **C**))

BGF

$$M(z,u) = \exp\Big(\ln \frac{1}{1-uC(z)}\Big) = \frac{1}{1-uC(z)}$$

Expected # nodes on cycles

$$\frac{N!}{N^N}[z^N]\frac{\partial}{\partial u}M(z,u)|_{u=1} = \frac{N!}{N^N}[z^N]\frac{C(z)}{(1-C(z))^2}$$

$$\sim \frac{N!}{N^N}[z^N]\frac{1}{2}\frac{1}{1-ez}$$

$$= \frac{1}{2}\frac{N!e^N}{N^N}$$

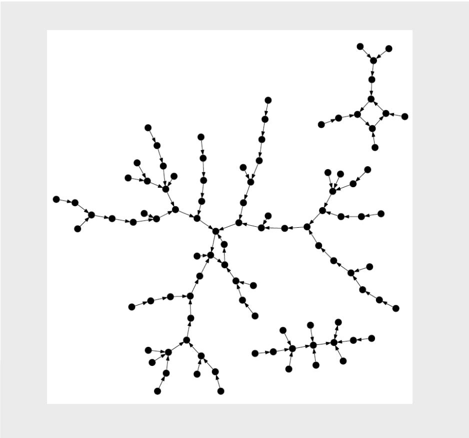
$$\sim \sqrt{\pi N/2}$$

$$C(z) \sim 1 - \sqrt{2}\sqrt{1-ez}$$

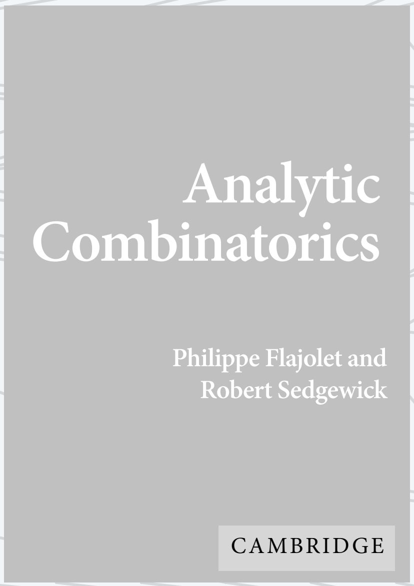
$$\frac{C(z)}{(1-C(z))^2} \sim \frac{1}{2}\frac{1}{1-ez}$$

Stirling

$$N! \sim \sqrt{2\pi N}\Big(\frac{N}{e}\Big)^N$$



predicted: 12.5
actual: 9



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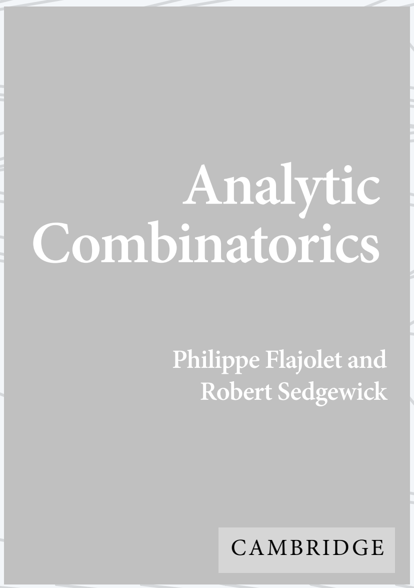
7. Applications of Singularity Analysis

- Simple varieties of trees
- Labelled sets
- **Mappings**
- Tree-like classes



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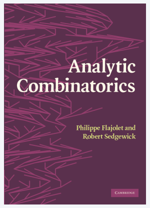
7. Applications of Singularity Analysis

- Simple varieties of trees
- Labelled sets
- Mappings
- **Tree-like classes**

Schema example 4: Implicit tree-like classes

Definition. A combinatorial class whose enumeration GF satisfies $F(z) = \Phi(z, F(z))$ is said to be an *implicit tree-like class* with *characteristic function G.*



page 467

unlabelled case: number of structures is $[z^N]F(z)$

F = CONSTRUCT(**Z, F**)
where CONSTRUCT is an arbitrary composition of +, ×, and SEQ

labelled case: number of structures is $N![z^N]F(z)$

F = CONSTRUCT(**Z, F**)
where CONSTRUCT is an arbitrary composition of +, ★, SEQ, SET, and CYC

$F(z) = \Phi(z, F(z))$

immediate via symbolic transfer

Example: Simple varieties of trees

$$\Phi(z, w) = z\phi(w)$$
$$F(z) = z\phi(F(z))$$

Smooth-implicit-function tree-like classes

smooth implicit function: A technical condition that enables us to unify the analysis of tree-like classes.

Definition. *Smooth-implicit-function tree-like classes.*

A tree-like class **F** = CONSTRUCT(**F**) with enumerating GF $F(z) = \Phi(z, F(z))$ is said to be *smooth-implicit*(r, s) if its characteristic function $\Phi(z, w)$ satisfies the following conditions:

- $\Phi(z, w)$ is analytic at 0 and in a domain $|z| < R$ and $|w| < S$ for some $R, S > 0$.
- $[z^N w^k]\Phi(z, w) \geq 0$ and >0 for some N and some $k > 2$, with $\Phi(0, 0) \neq 0$.
- There exist positive reals $r < R$ and $s < S$ such that $\Phi(r, s) = s$ and $\Phi_w(r, s) = 1$.

$\Phi(z, w) = w$

$\Phi_w(z, w) = 1$

"*characteristic system*"

Example: "phylogenetic trees" [*details to follow*]

Construction	$\mathbf{L} = \mathbf{Z} + SET_{\geq 2}(\mathbf{L})$
OGF equation	$L(z) = z + e^{L(z)} - 1 - L(z)$
Characteristic function	$\Phi(z, w) = z - 1 + e^w - w$
Characteristic system	$z + e^w - 1 - w = w$ $e^w - 1 = 1$

← *solution* $r = 2\ln 2 - 1$, $s = \ln 2$

phylogenetic trees are smooth-implicit($2\ln 2 - 1$, $\ln 2$)

Transfer theorem for implicit tree-like classes

Theorem. *Asymptotics of implicit tree-like classes.*

Suppose that **F** is an implicit tree-like class with characteristic function $\Phi(z, w)$ and aperiodic and smooth-implicit(r, s) GF $F(z) = \Phi(z, F(z))$, so that $\Phi(r, s) = s$ and $\Phi_w(r, s) = 1$.

Then $F(z)$ converges at $z = r$ where it has a square-root singularity with

$F(z) \sim s - \alpha\sqrt{1 - z/r}$ and $[z^N]F(z) \sim \dfrac{\alpha}{2\sqrt{\pi}}\left(\dfrac{1}{r}\right)^N N^{-3/2}$ where $\alpha = \sqrt{\dfrac{2r\Phi_z(r,s)}{\Phi_{ww}(r,s)}}$.

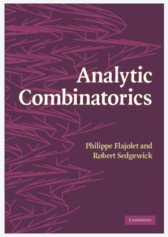
Example: binary trees (alternate)

Construction	$B = \bullet + \bullet \times SEQ_{0,2}(B)$
OGF equation	$B(z) = z + zB(z)^2$
Characteristic function	$\Phi(z, w) = z + w^2$
Characteristic system	$z + w^2 = w$ $2w = 1$
Coefficient asyptotics	$[z^N]B(z) \sim \dfrac{1}{\sqrt{\pi}} 4^N N^{3/2}$

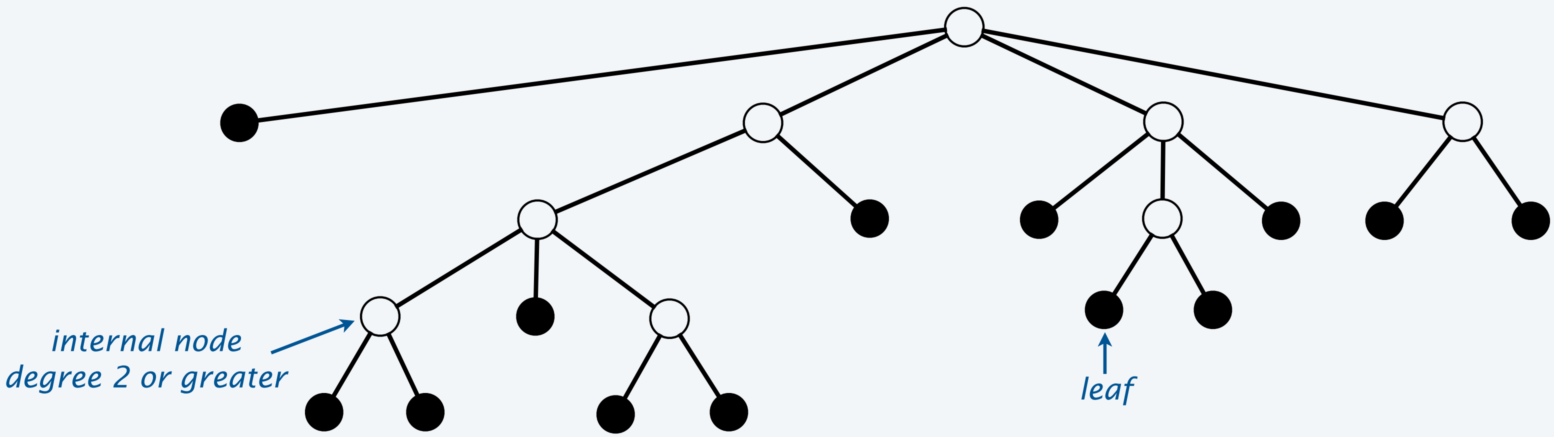
$$s = 1/2$$
$$r = 1/4$$
$$\Phi_z(z, w) = 1$$
$$\Phi_w(z, w) = 2w$$
$$\Phi_{ww}(z, w) = 2$$
$$\alpha = 2$$

Example 8. Bracketings

Def. A *bracketing* of *N* items is a tree with *N* leaves and no unary nodes



page 69



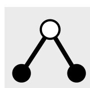
Applications

- Parenthesizations.
- Series-parallel networks.
- Schröder's 2nd problem

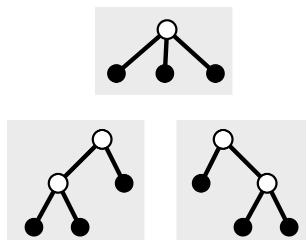
Example 8: Bracketings

Q. How many **bracketings** with N leaves?

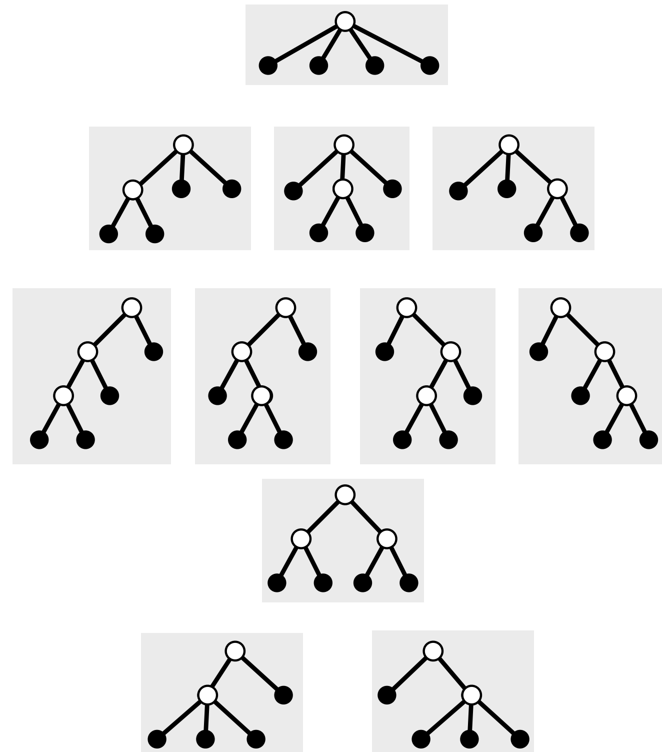
$S_1 = 1$



$S_2 = 1$



$S_3 = 3$



$S_4 = 11$

All nodes of degree 0 (leaves) or >1 (internal nodes)

size: number of leaves

Example 8: Bracketings

Q. How many parenthesizations of N items?

a

$S_1 = 1$

(a b)

$S_2 = 1$

(a b c)

((a b) c) (a (b c))

$S_3 = 3$

(a b c d)

((a b) c d) (a (b c) d) ((a b) c d)

(((a b) c) d) ((a (b c)) d) (a ((b c) d)) (((a b) c) d)

((a b) (c d))

((a b c) d) (a (b c d))

$S_4 = 11$

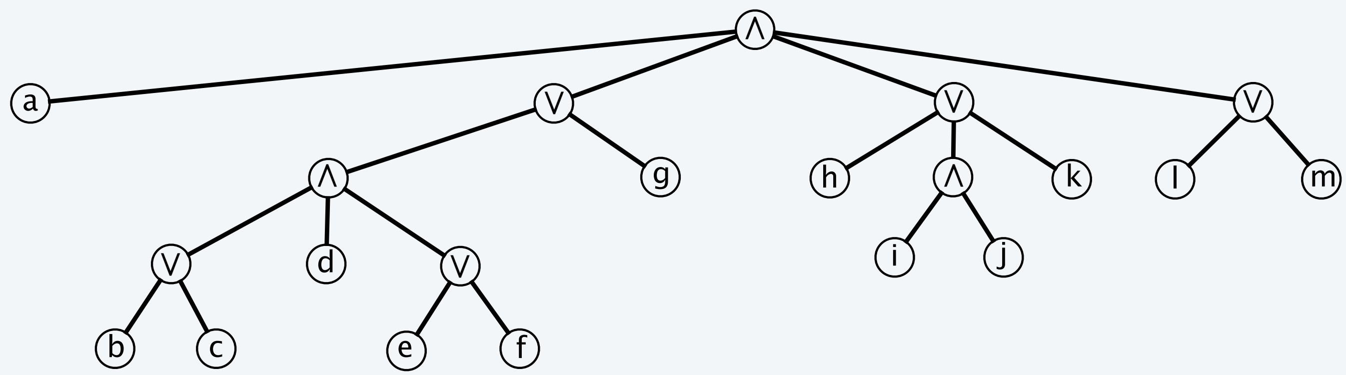
Example 8: Bracketings

Three additional equivalent structures.

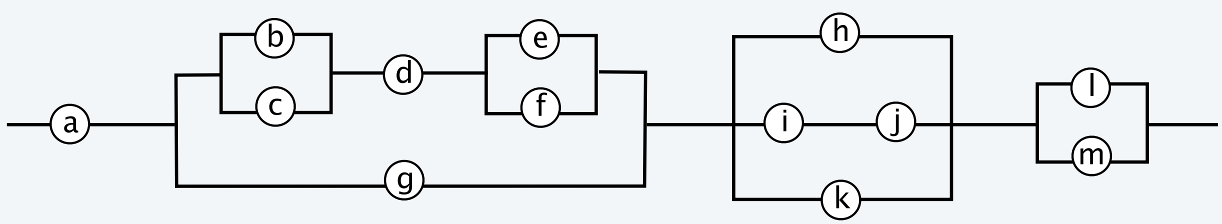
and-or conjunctive propositions

$a \wedge ((b \vee c) \wedge d \wedge (e \vee f) \vee g) \wedge (h \vee (i \wedge j) \vee k) \wedge (l \vee m)$

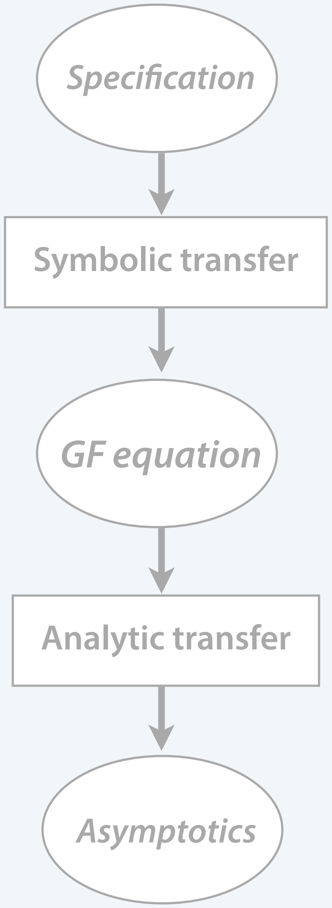
and-or trees



series-parallel networks



Example 8: Bracketings



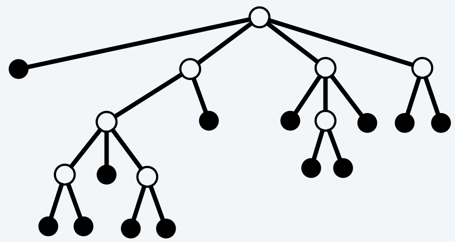
S, the class of all bracketings

$$\mathbf{S} = \mathbf{Z} + SEQ_{>1}(\mathbf{S})$$

$$S(z) = z + \frac{1}{1 - S(z)} - 1 - S(z)$$

$$[z^N]S(z) \sim \sqrt{\frac{r}{8\sqrt{2\pi}}}\left(\frac{1}{r}\right)^N N^{-3/2}$$

with $r = 3 - 2\sqrt{2}$



Note that the specification is the *most succinct* of all the descriptions

Theorem. *Asymptotics of implicit tree-like classes.*

Suppose that **F** is an implicit tree-like class with characteristic function $\Phi(z, w)$ and aperiodic and smooth-implicit(r, s) GF $F(z) = \Phi(z, F(z))$, so that $\Phi(r, s) = s$ and $\Phi_w(r, s) = 1$.

Then $F(z)$ converges at $z = r$ where it has a square-root singularity with

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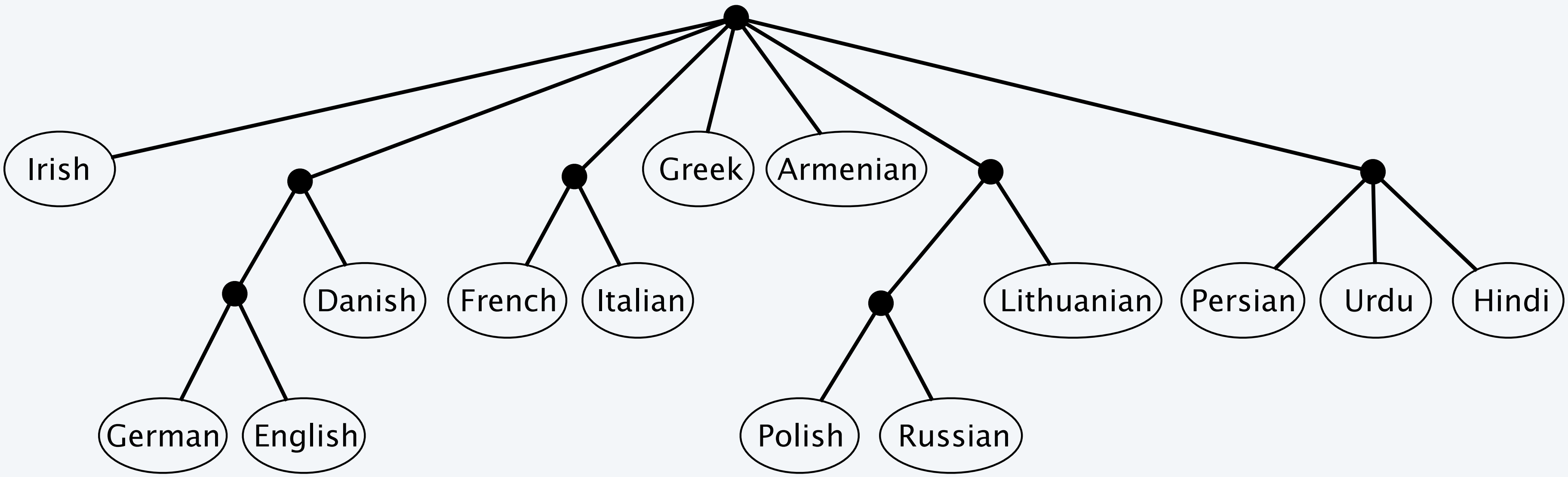
[*details left for exercise*]

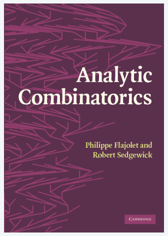
Example 9. Labelled hierarchies (phylogenetic trees)

Def. A *labelled hierarchy* of *N* items is a tree with N labelled leaves and no unary nodes



Applications

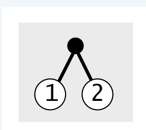
- Classification.
- Evolution of genetically related organisms.
- Schröder's 4th problem



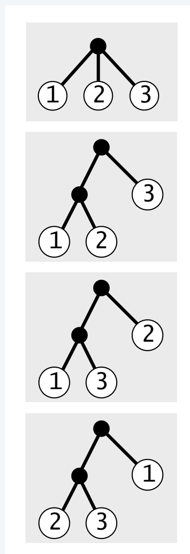
page 128

Example 9. Labelled hierarchies (phylogenetic trees)

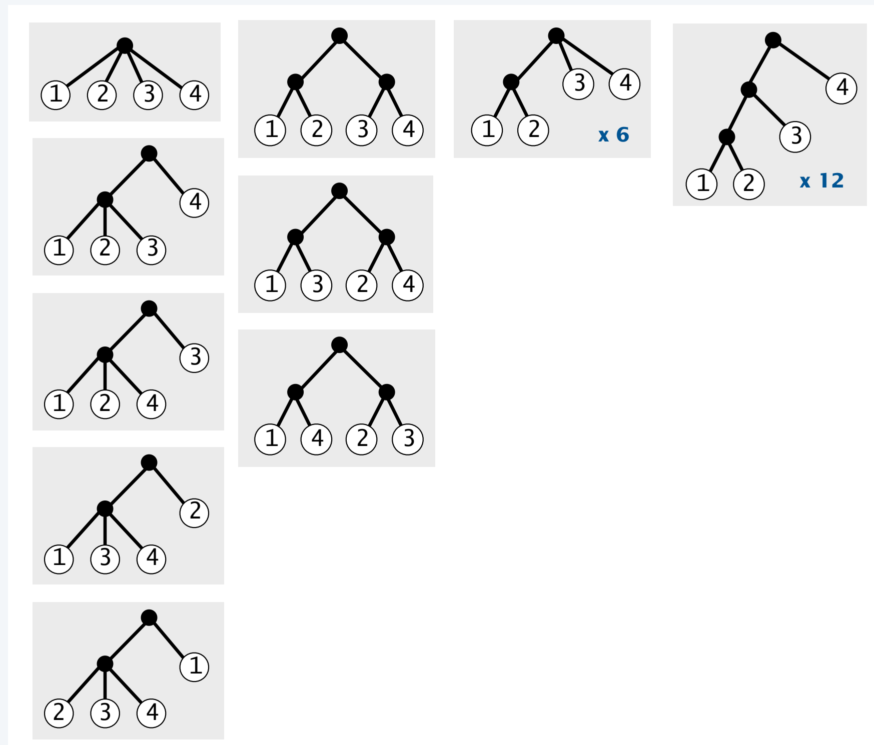
Q. How many different *labelled hierarchies* of N nodes?



$L_2 = 1$

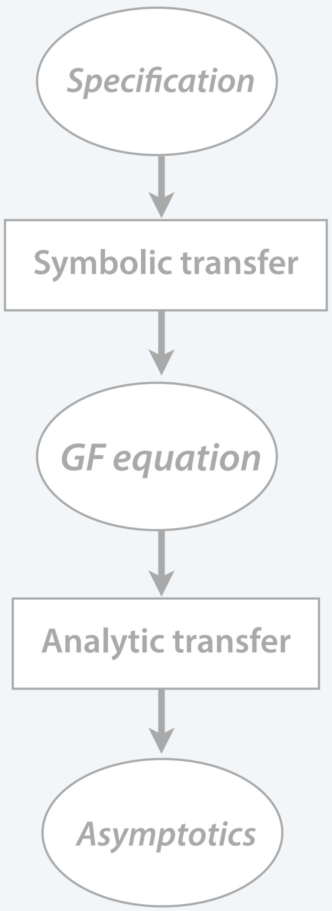


$L_3 = 4$



$L_4 = 26$

Example 9. Labelled hierarchies (phylogenetic trees)



L, the class of labelled hierarchies

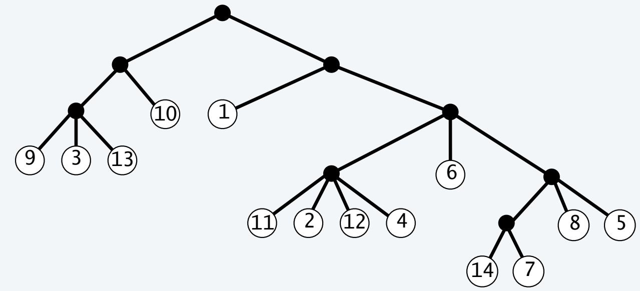
L = **Z** + $SET_{\geq 2}$(**L**)

$$L(z) = z + e^{L(z)} - 1 - L(z)$$

implicit tree-like

$$N![z^N]L(z) \sim N! \frac{\sqrt{r}}{2\sqrt{\pi N^3}} \Big(\frac{1}{r}\Big)^N N^{3/2}$$

with $r = 2\ln 2 - 1$



Theorem. *Asymptotics of implicit tree-like classes.*

Suppose that **F** is an implicit tree-like class with characteristic function $\Phi(z, w)$ and aperiodic and smooth-implicit(r, s) GF $F(z) = \Phi(z, F(z))$, so that $\Phi(r, s) = s$ and $\Phi_w(r, s) = 1$.

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$$z + e^w - 1 - w = w \qquad r = 2\ln 2 - 1$$

$$e^w - 1 = 1 \qquad s = \ln 2$$

$$\Phi(z, w) = z - 1 + e^w - w$$

$$\Phi_z(z, w) = 1$$

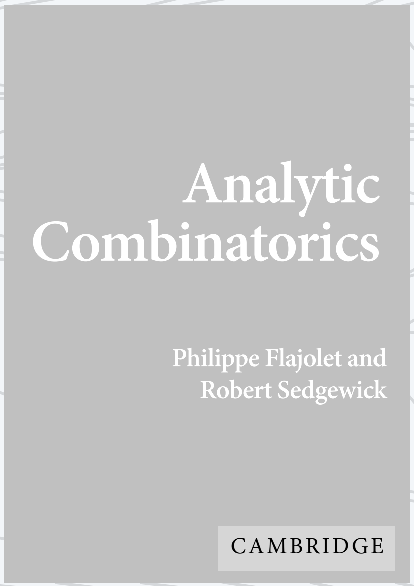
$$\Phi_w(z, w) = e^w - 1$$

$$\Phi_{ww}(z, w) = e^w$$

$$\Phi_z(r, s) = 1$$

$$\Phi_{ww}(r, s) = 2$$

$$\alpha = \sqrt{2\ln 2 - 1}$$



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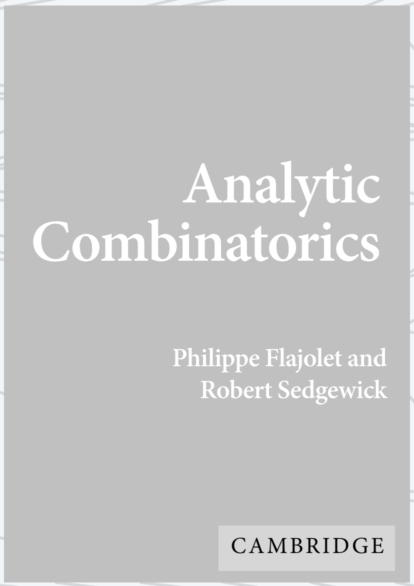
7. Applications of Singularity Analysis

- Simple varieties of trees
- Labelled sets
- Mappings
- **Tree-like classes**



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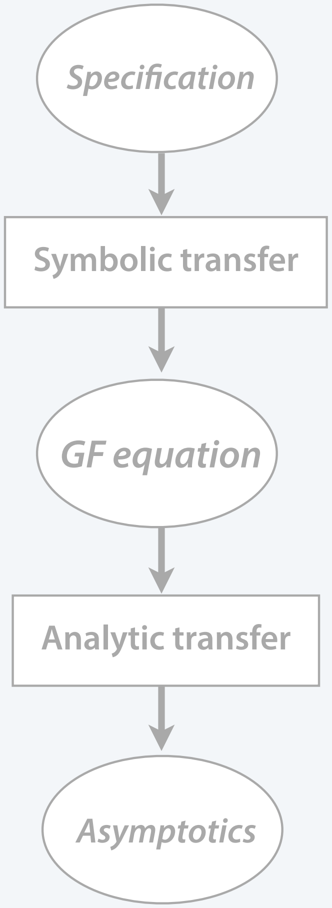
7. Applications of Singularity Analysis

- Set schema
- Simple varieties of trees
- Mappings
- Tree-like classes
- **Summary**

Singularity analysis: examples of applications

	construction	*generating function*	*coefficient asymptotics*
rooted ordered trees	G = Z × SEQ(G)	$G(z) = \frac{z}{1 - G(z)}$	$\frac{1}{4\sqrt{\pi}} 4^N N^{3/2}$
binary trees	B = • × (E + B) × (E + B) B = • + • × SEQ0,2(B)	$B(z) = z(1 + B(z)^2)$ $B(z) = z + zB(z)^2)$	$\frac{1}{\sqrt{\pi}} 4^N N^{3/2}$
unary-binary trees	M = • × SEQ0,1,2(M)	$M(z) = z(1 + M(z) + M(z)^2)$	$\frac{1}{\sqrt{4\pi/3}} 3^N N^{-3/2}$
Cayley trees	C = Z ★ SET(C)	$C(z) = ze^{C(z)}$	$N! \frac{1}{\sqrt{2\pi}} e^N N^{-3/2} = N^{N-1}$
mapping components	K = CYC(C)	$K(z) = \ln \frac{1}{1 - C(z)}$	$\sim N! \frac{e^N}{2N} \sim \sqrt{\frac{\pi}{2N}} N^N$
mappings	M = SET(K)	$M(z) = e^{K(z)} = \frac{1}{1 - C(z)}$	$\sim N! \frac{e^N}{\sqrt{2\pi N}} \sim N^N$
2-regular graphs	R = SET(UCYC>2 (Z))	$R(z) = \frac{e^{-z/2 - z^2/4}}{\sqrt{1 - z}}$	$\sim N! \frac{e^{-3/4}}{\sqrt{\pi N}}$
labelled hierarchies	L = Z + SET≥2(L)	$L(z) = z + e^{L(z)} - 1 - L(z)$	$\frac{\sqrt{2 \ln 2 - 1}}{2\sqrt{\pi N^3}} \frac{N!}{(2 \ln 2 - 1)^N}$

"If you can specify it, you can analyze it"

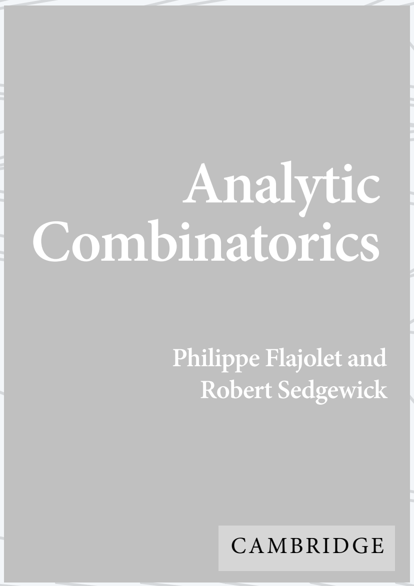


Singularity analysis is an effective approach for analytic transfer from GF equations to coefficient asymptotics for classes *with GFs that are not meromorphic*.

Schema can unify the analysis for entire families of classes.

schema	*technical condition*	*construction*	*coefficient asymptotics*
Labelled set	exp-log	**F** = SET(**G**)	$\frac{e^{\beta}}{\Gamma(\alpha)}\left(\frac{1}{\rho}\right)^{N} N^{1-\alpha}$
Simple variety of trees	invertible	**F** = **Z** × *SEQ*(**F**) **F** = **Z** ★ *SEQ*(**F**)	$\frac{1}{\sqrt{\alpha\pi}}\left(\frac{1}{\rho}\right)^{N} N^{-3/2}$
Context-free	irreducible	Family of (+, ×) constructs	$\frac{1}{\sqrt{\alpha\pi}}\left(\frac{1}{\rho}\right)^{N} N^{-3/2}$
Implicit tree-like	smooth implicit function	**F** = *CONSTRUCT*(**F**)	$\frac{\alpha}{2\sqrt{\pi}}\left(\frac{1}{r}\right)^{N} N^{-3/2}$

Next: GFs with no singularities.



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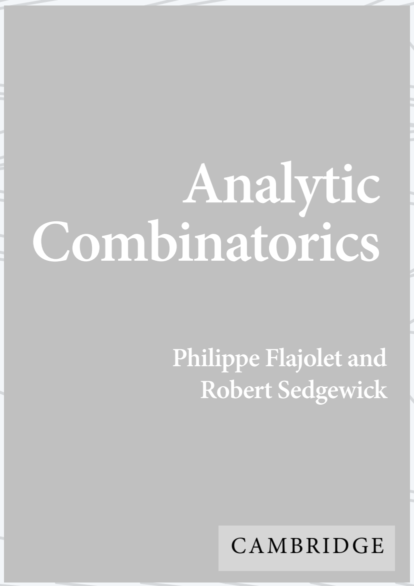
7. Applications of Singularity Analysis

- Set schema
- Simple varieties of trees
- Mappings
- Tree-like classes
- **Summary**



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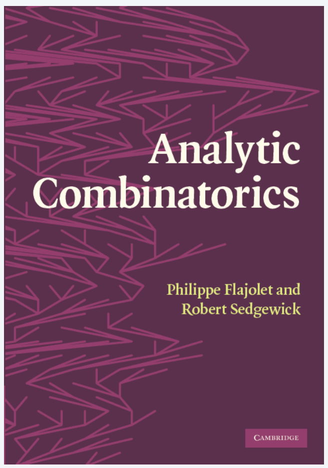
7. Applications of Singularity Analysis

- Set schema
- Simple varieties of trees
- Mappings
- Tree-like classes
- **Exercises**

Web Exercise VII.1

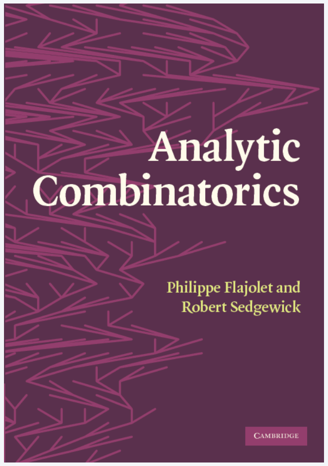
Bracketings (Schröder's 2nd problem)



Web Exercise VII.1. Use the tree-like schema to develop an asymptotic expression for the number of bracketings with *N* leaves (see Example I.15 on page 69 and Note VII.19 on page 474).

Assignments

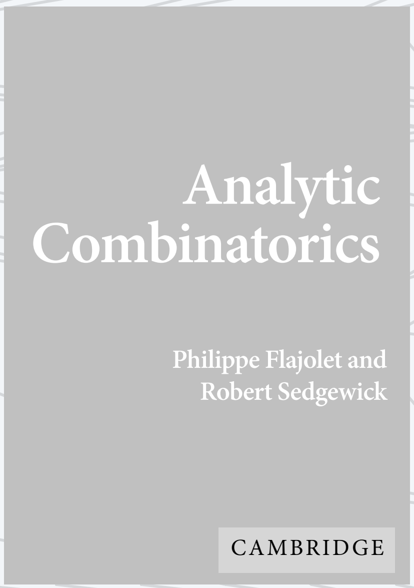
1. Read pages 439-540 (*Applications of Singularity Analysis*) in text.
   Usual caveat: Try to get a feeling for what's there, not understand every detail.



2. Write up a solutions to Web Exercise VII.1.

3. Programming exercise.

Program VII.1. Do *r*- and θ-plots of the GF for bracketings (see Web Exercise VII.1).



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7. Applications of Singularity Analysis

- Set schema
- Simple varieties of trees
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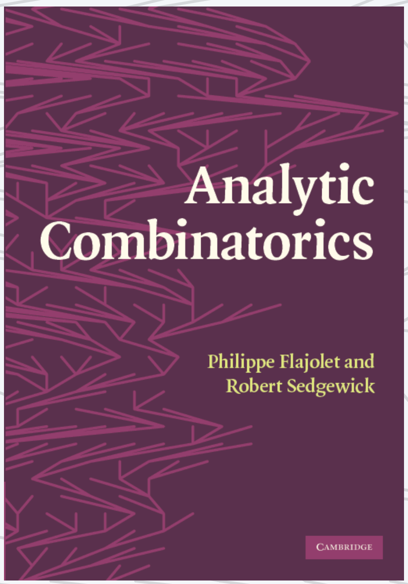
- **Exercises**



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7. Applications of Singularity Analysis