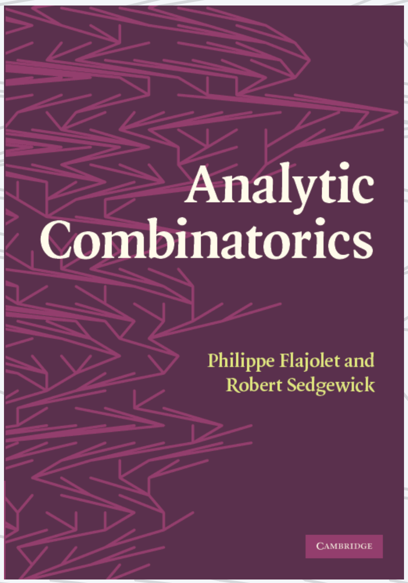
ANALYTIC COMBINATORICS

PART TWO

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# 8. Saddle-Point Asymptotics

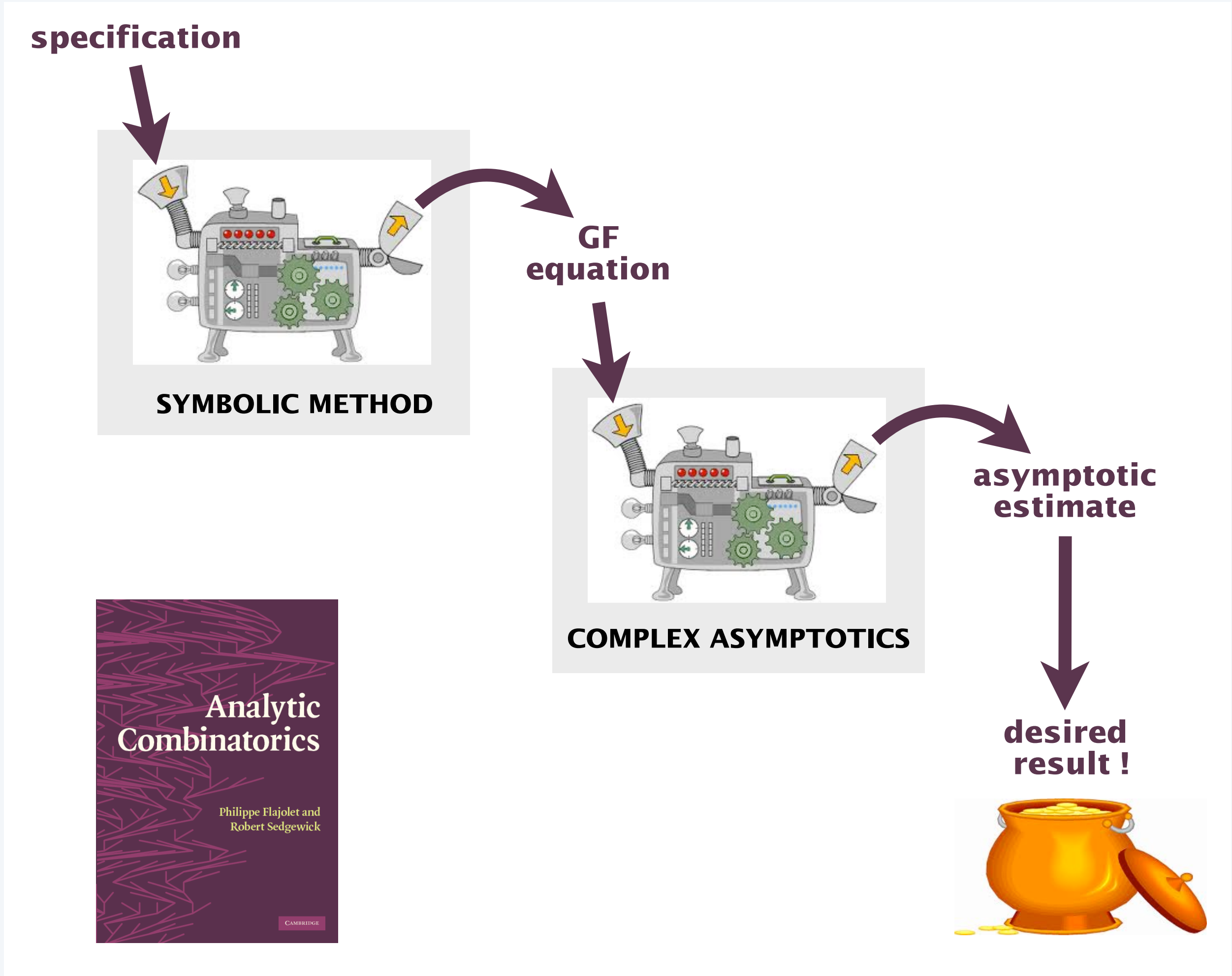
## Analytic combinatorics overview

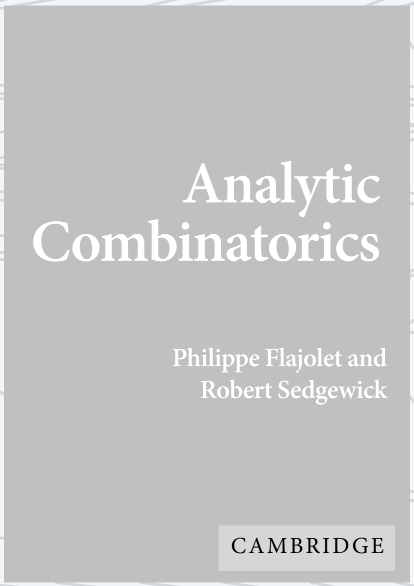
**A. SYMBOLIC METHOD**

1. OGFs
2. EGFs
3. MGFs

**B. COMPLEX ASYMPTOTICS**

4. Rational & Meromorphic
5. Applications of R&M
6. Singularity Analysis
7. Applications of SA
8. Saddle point



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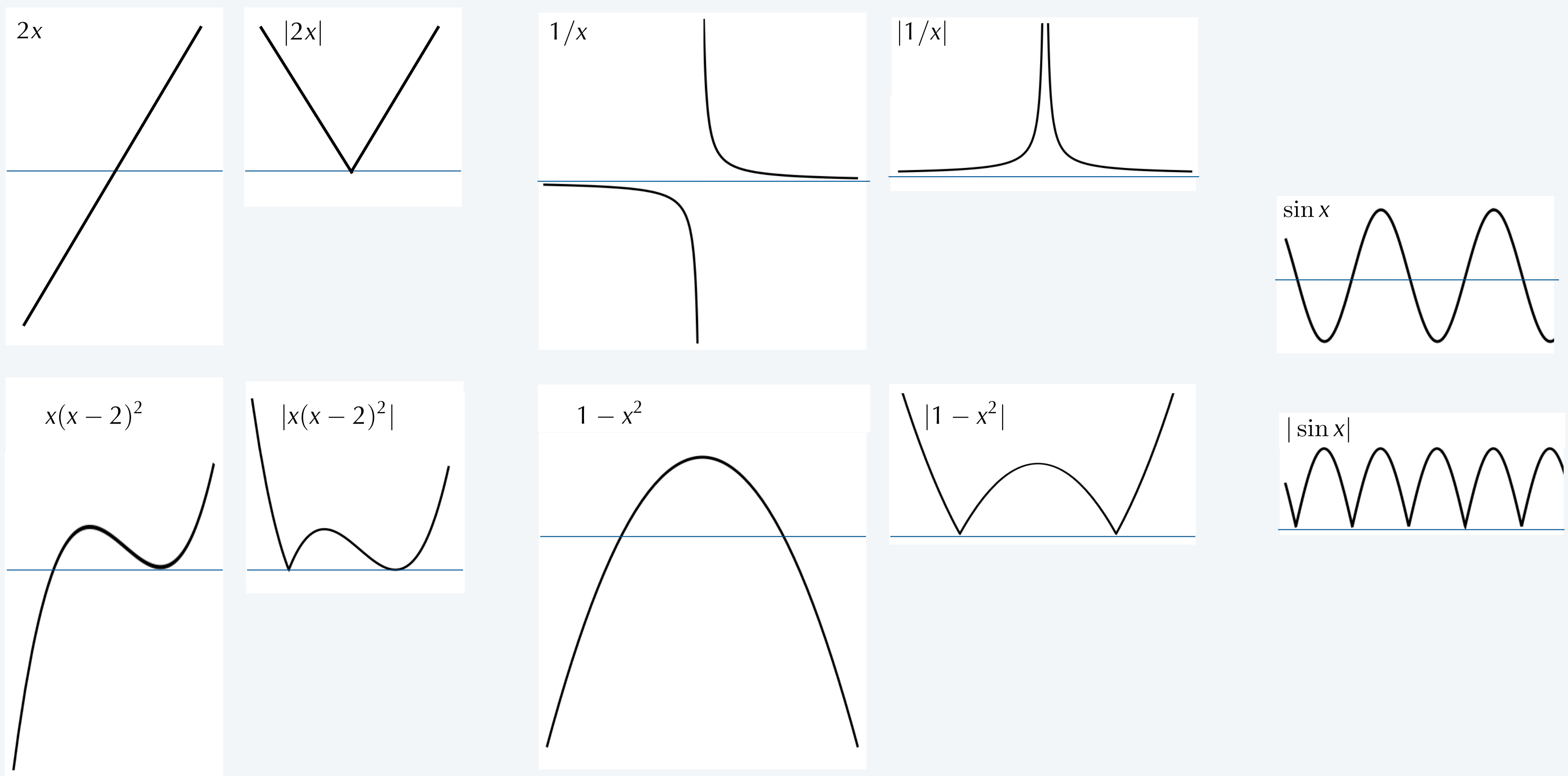
# 8. Saddle-Point Asymptotics

- **Modulus surfaces**
- Saddle point bounds
- Saddle point asymptotics
- Applications

# Warmup: 2D absolute value plots

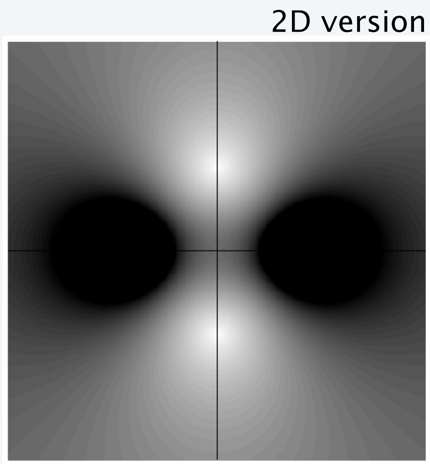
Consider 2D plots of functions: all points ($x$, $|f(x)|$ ) in a Cartesian plot.

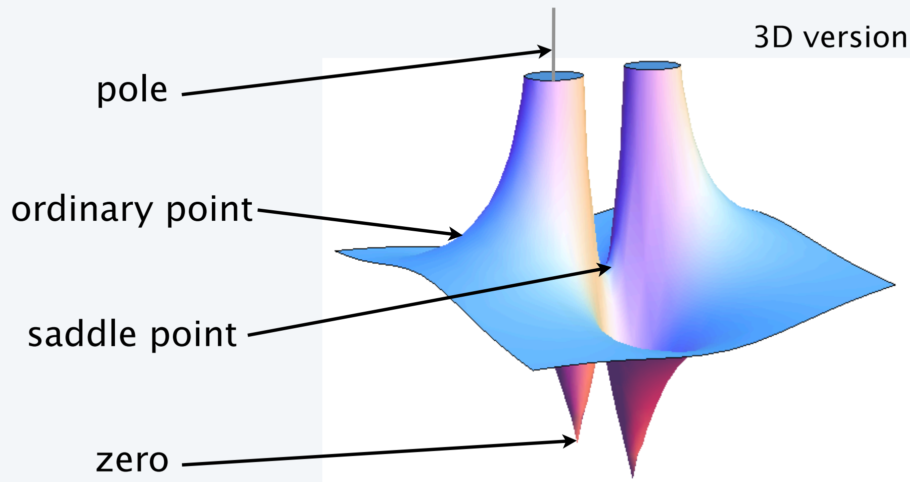


## Welcome to absolute-value-land!

Consider 3D versions of our plots of analytic functions.
A *modulus surface* is a plot of $(x, y, |f(z)|)$ where $z = x + yi$ .

Example: $\dfrac{1+4z^2}{1-4z^2}$




Q. Can a modulus surface assume any shape ?

A. No.

A. (A surprise.) Only four types of points.

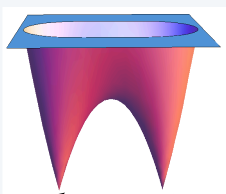
# Modulus surface points type I: zeros

A *zero* is a point where $f(z) = 0$ and $f'(z) \neq 0$.

Key point: All zeros have the same local behavior.

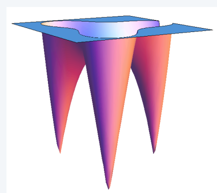
$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \ldots \quad \sim f'(z_0)(z - z_0)$$

$1 - z^2$

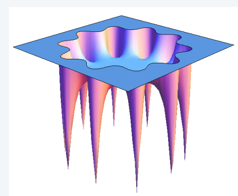


$\sim 2(z+1)$ $\qquad$ $\sim -2(z-1)$

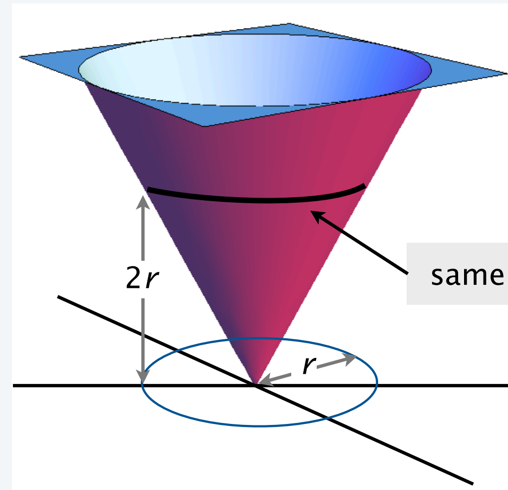
$1 - z^3$



$1 - z^8$



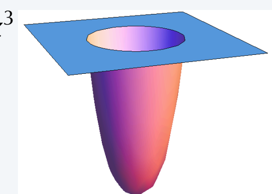
Ex. $f(z) = 2z = 2re^{i\theta}$, $|f(z)| = 2r$




same for all θ

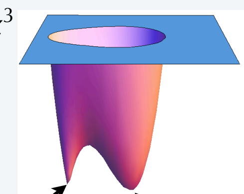
A *zero of order p* is a point where $f^{(k)}(z) = 0$ for $0 \leq k < p$ and $f^{(p)}(z) \neq 0$.

$z^3$



zero of order 3

$z^2 + z^3$

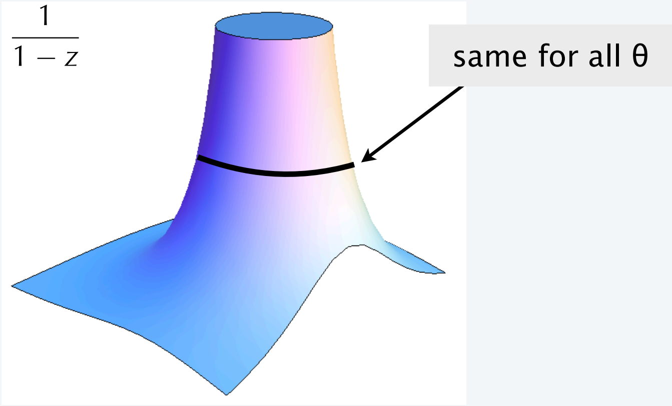


zero (order 1)

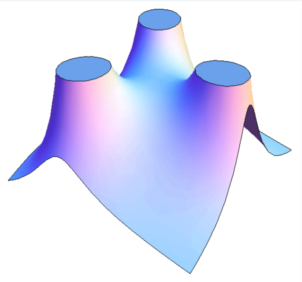
zero of order 2

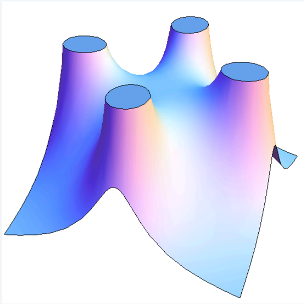
## Modulus surface points type II: poles

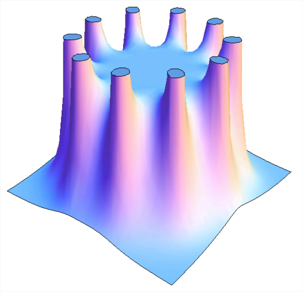
A *pole* is a point $z_0$ where $f(z) \sim \dfrac{c}{z - z_0}$

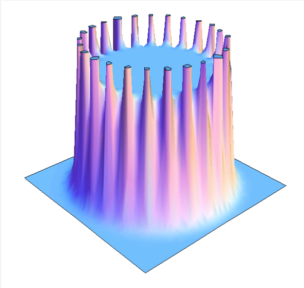



By definition, all poles have the same local behavior.

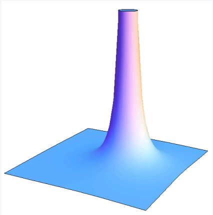






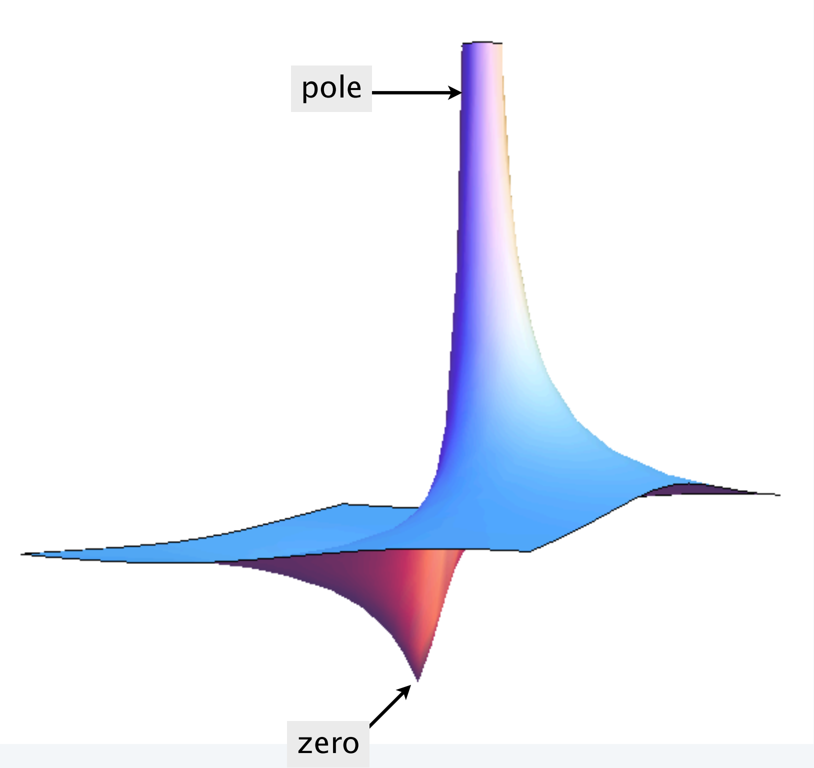


A *pole of order p* is a point $z_0$ where $f(z) \sim \dfrac{c}{(z - z_0)^p}$



## Quick in-class exercise

Q. What function is this?



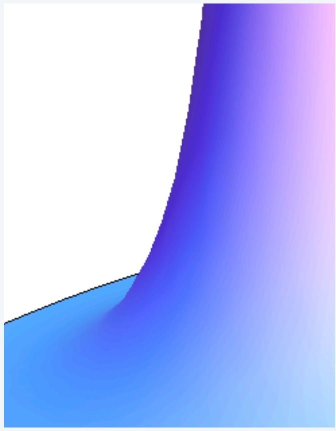

A. $\dfrac{z}{1-z}$

# Modulus surface points type III: ordinary points

An *ordinary point* is a point where $f(z) \neq 0$ and $f'(z) \neq 0$.

All ordinary points have the same local behavior.

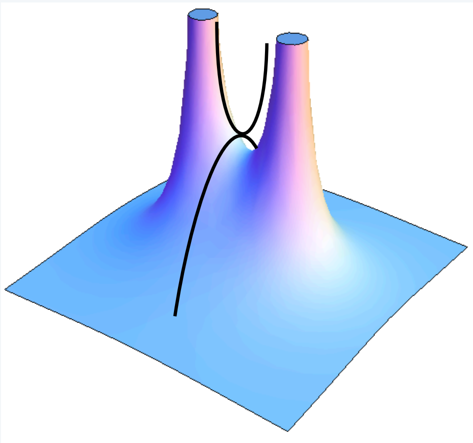
$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \ldots \quad \boxed{\sim c}$$



## Modulus surface points type III: saddle points

A *saddle point* is a point where $f(z) \neq 0$ and $f'(z) = 0$.

$$\frac{1}{(1-z)(2-z)}$$



All saddle points have the same local behavior.

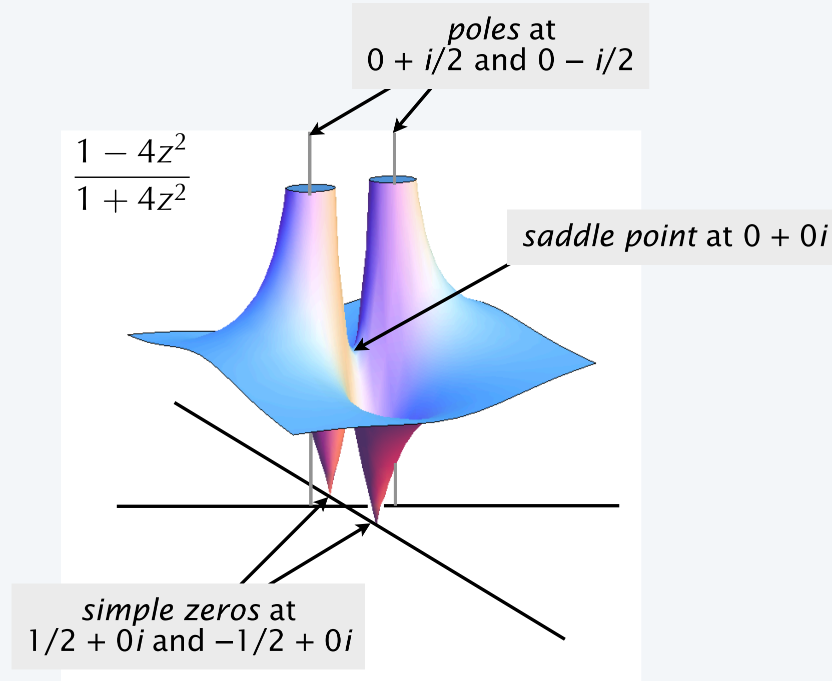
$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \ldots \quad \sim c(z - z_0)^2$$

Basic characteristic

- Downwards-oriented parabola at one angle
- Upwards-oriented parabola at perpendicular angle

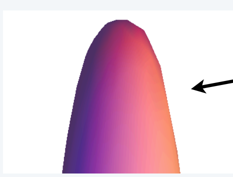
# Modulus surface points: summary

| | $f(z)$ | $f'(z)$ | *local behavior* |
|---|---|---|---|
| simple zero | 0 | *not* 0 | $\sim c\,(z - z_0)$ |
| zero of order $p > 1$ | 0 | 0 | $\sim c\,(z - z_0)^p$ |
| saddle point | *not* 0 | 0 | $\sim c\,(z - z_0)^2$ |
| ordinary point | *not* 0 | *not* 0 | $\sim c$ |
| simple pole | | | $\sim c\,/\,(z - z_0)$ |




$$\left(\frac{1-4z^2}{1+4z^2}\right)' = \frac{-8z(1+4z^2) - (1-4z^2)8z}{(1+4z^2)^2} = -\frac{16z}{(1-4z^2)^2}$$

*Maximum modulus principle*: There are no other possibilities (!)



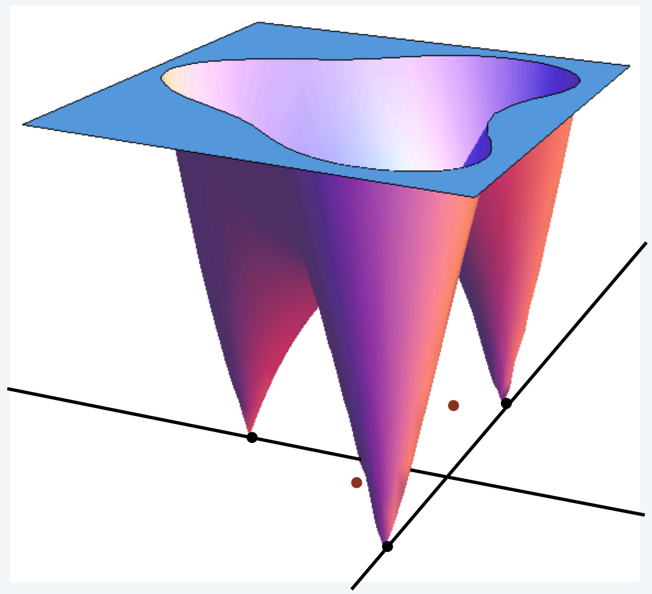
Example: No local maxima

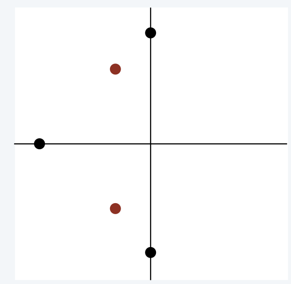
## Quick in-class exercise

Q. Where are the saddle points?

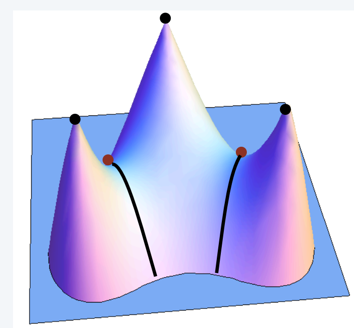
- zeros ( $-1$, $-i$, $+i$)
- saddle points

$$1 + z + z^2 + z^3$$





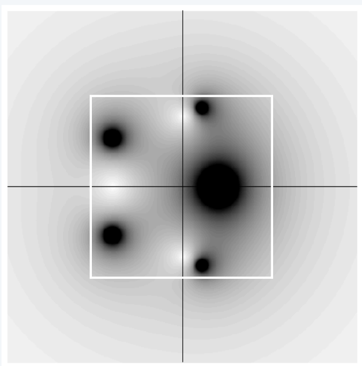
bottom view

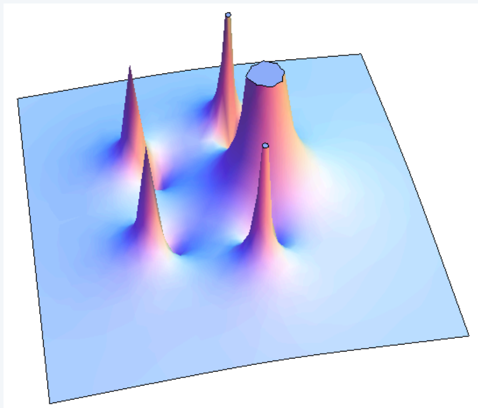


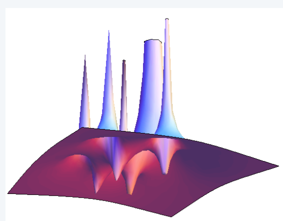
A. Where $1 + 2z + 3z^2 = 0$ , or $z = -\dfrac{1}{3} \pm \dfrac{i\sqrt{2}}{3}$

## Modulus surface plots for familiar AC GFs

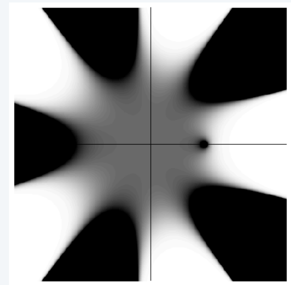
$$\frac{1+z+z^2+z^3+z^4}{1-z-z^2-z^3-z^4-z^5}$$

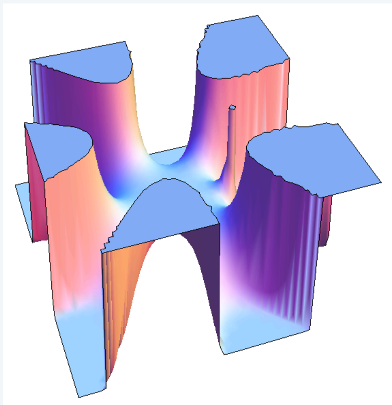




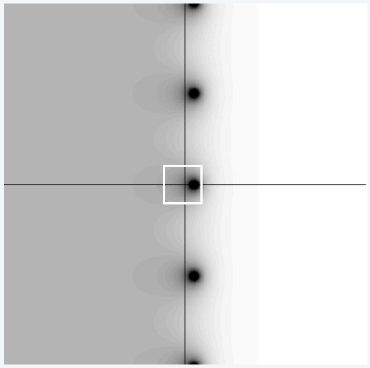


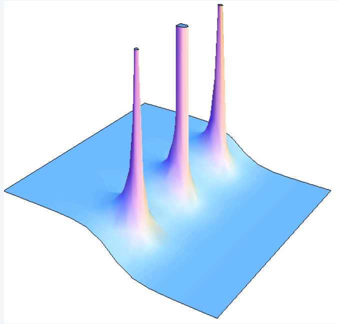
$$\frac{e^{-z-z^2/2-z^3/3-z^4/4-z^5/5}}{1-z}$$

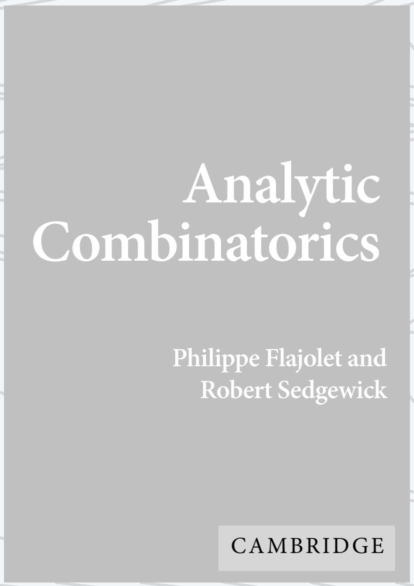




$$\frac{1}{2-e^z}$$





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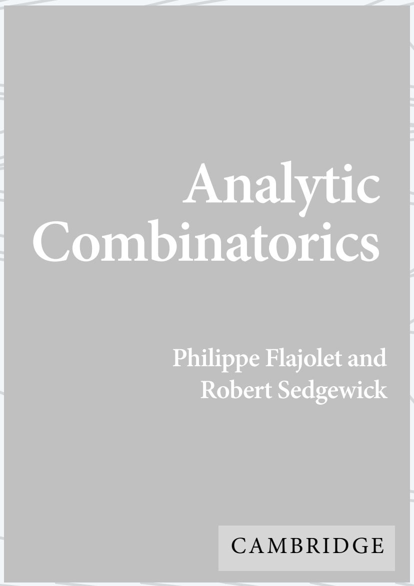
# 8. Saddle-Point Asymptotics

- **Modulus surfaces**
- Saddle point bounds
- Saddle point asymptotics
- Applications

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## Saddle-point bound for GFs: basic idea

Cauchy coefficient formula

$$[z^N]G(z) = \frac{1}{2\pi i}\int_C G(z)\frac{dz}{z^{N+1}}$$

Saddle point bound:

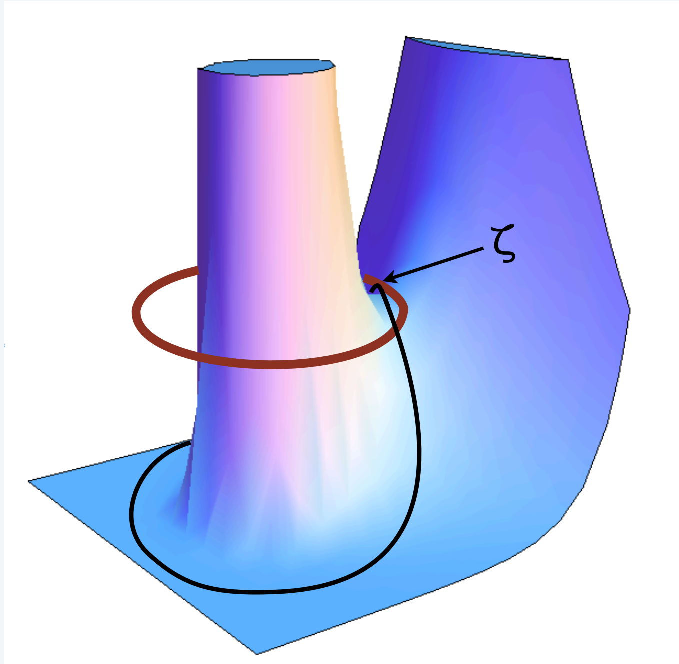
- Saddle point at ζ ← "zeta"
- Use circle of radius ζ
- Integrand is ≤ $G(\zeta)/\zeta^{N+1}$ everywhere on circle

Note: ζ is the solution to $\left(\frac{G(z)}{z^{N+1}}\right)' = 0$

$$\frac{G'(z)}{z^{N+1}} - (N+1)\frac{G(z)}{z^{N+2}} = 0$$

$$zG'(z)/G(z) = N+1$$ ← "saddle point equation"

Example: $e^z/z^6$



## Saddle-point bounds for GFs

**Theorem.** *Saddle point bounds for GFs.*

Let $G(z)$, not a polynomial, be analytic at the origin with finite radius of convergence $R$.

If G has nonnegative coefficients, then $[z^N]G(z) \leq G(\zeta)/\zeta^N$ where $\zeta$ is the *saddle point* closest to the origin, the unique real root of the *saddle point equation* $\zeta G'(\zeta)/G(\zeta) = N+1$.

**Proof (sketch).** By Cauchy coefficient formula

$$[z^N]G(z) = \frac{1}{2\pi i}\int_C G(z)\frac{dz}{z^{N+1}}$$

$$= \frac{\zeta}{2\pi}\int_0^{2\pi} G(z)\frac{d\theta}{z^{N+1}}$$

Take $C$ to be a circle of radius $\zeta$ and change to polar coordinates

$$\leq \frac{G(\zeta)}{\zeta^N}$$

$G(z) \leq G(\zeta)/\zeta^{N+1}$ on $C$

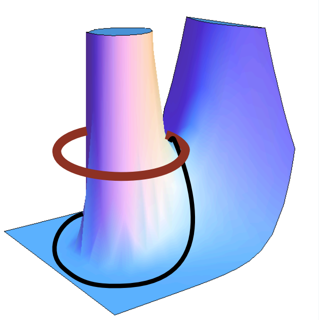
Example: $[z^5]e^z$

$$G(z) = e^z$$
$$G'(z) = e^z$$
$$\zeta = 6$$

$$[z^N]e^z = \frac{1}{5!} \leq \frac{e^6}{6^5}$$

$\doteq .008333$ $\quad$ $\doteq .009498$

$e^z/z^6$



## Saddle point GF bound example I: factorial/exponential

Goal. Estimate $\frac{1}{N!} = [z^N]e^z$

$$G(z) = e^z$$

Saddle point equation $\quad z\frac{e^z}{e^z} = N+1$

Saddle point $\quad \zeta = N+1$

Saddle point bound $\quad [z^N]e^z = \frac{1}{N!} \le \frac{e^{N+1}}{(N+1)^N}$

$$\to \frac{e^N}{N^N}$$

Saddle point equation

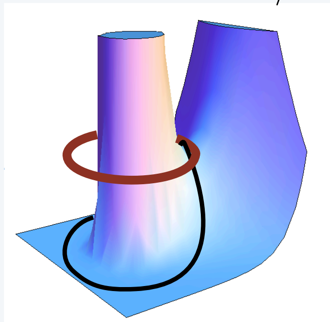
$$z\frac{G'(z)}{G(z)} = N+1$$

Saddle point bound

$$[z^N]G(z) \le G(\zeta)/\zeta^N$$

$$(1+\frac{1}{N})^N \to e$$

$e^z/z^6$



Bound is too high by only a factor of $\sqrt{2\pi N}$, since $\frac{1}{N!} \sim \frac{e^N}{N^N\sqrt{2\pi N}}$

## Saddle point GF bound example II: Catalan/central binomial

Goal. Estimate $\binom{2N}{N} = [z^N](1+z)^{2N}$

$$G(z) = (1+z)^{2N}$$

Saddle point equation $\quad 2Nz = (N+1)(1+z)$

Saddle point $\quad \zeta = \dfrac{N+1}{N-1}$

Saddle point bound $\quad \binom{2N}{N} \le \dfrac{\left(\frac{2N}{N-1}\right)^{2N}}{\left(\frac{N+1}{N-1}\right)^{N}} = \left(\dfrac{4N^2}{N^2-1}\right)^N$

$$\to 4^N$$

Saddle point equation

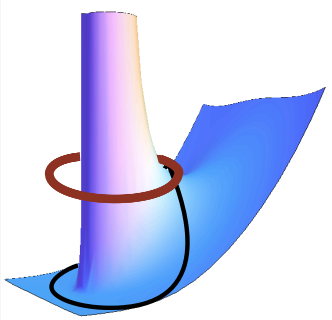
$$z\frac{G'(z)}{G(z)} = N+1$$

$$z\frac{2N(1+z)^{2N-1}}{(1+z)^{2N}} = N+1$$

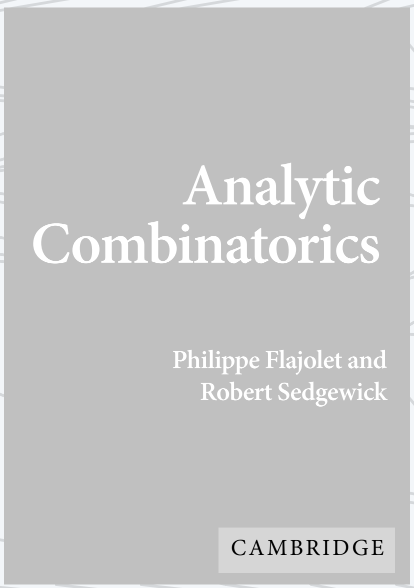
Saddle point bound

$$[z^N]G(z) \le G(\zeta)/\zeta^N$$

$(1+z)^{10}/z^6$



Bound is too high by only a factor of $\sqrt{\pi N}$, since $\binom{2N}{N} \sim \dfrac{4^N}{\sqrt{\pi N}}$

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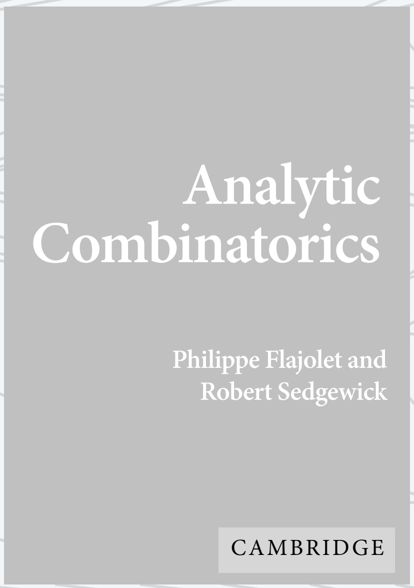


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- Modulus surfaces
- Saddle point bounds
- **Saddle point asymptotics**
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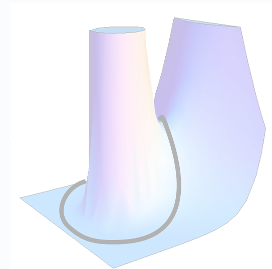
# Saddle-point *method* for GFs: basic idea

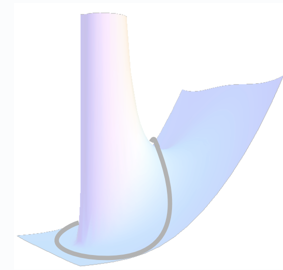
$e^z/z^6$

$(1+z)^{10}/z^6$

Cauchy coefficient formula

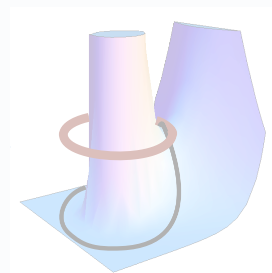
$$[z^N]G(z) = \frac{1}{2\pi i}\int_C G(z)\frac{dz}{z^{N+1}}$$

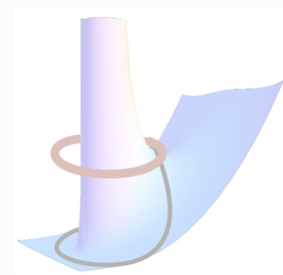




Saddle point bound:

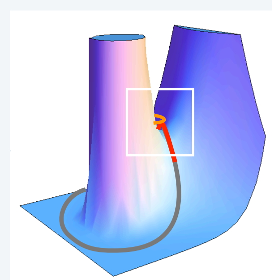
- Saddle point at $\zeta$
- Use circle of radius $\zeta$
- Integrand is $\leq G(\zeta)/\zeta^{N+1}$ everywhere on circle

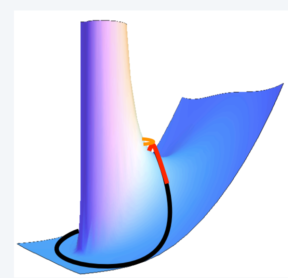




Saddle point *method*:

- Focus on path near saddle point
- Bound "tail" contribution
- Use Laplace's method





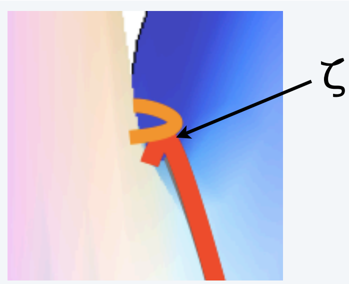
## Saddle-point susceptibility

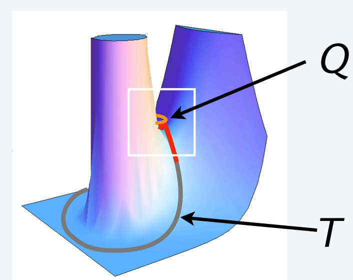
*susceptibility*: Technical conditions that enable us to unify saddle-point approximations.

Definition. *Saddle-point susceptible contour integrals.*

The contour integral $\int_C F(z)dz$ with $F(z) = e^{f(z)}$ is *susceptible to the saddle point approximation* if $C$ passes through a saddle point ζ, the unique real root of the saddle point equation $F'(z) = 0$ (or $f'(z) = 0$) and $C$ can be split into two parts $T$ and $Q$ such that

- Tails are negligable: $\int_T F(z)dz = o\Big(\int_C F(z)dz\Big)$
- A central quadratic approximation holds uniformly along $Q$: $f(z) \sim f(\zeta) + \dfrac{1}{2}f''(\zeta)(z-\zeta)^2$
- Tails can be completed back [details omitted].




to be expected unless multiple saddle point since f '(ζ)= 0

## Saddle-point transfer theorem

**Theorem.** If a contour integral $\int_C F(z)dz$ with $F(z) = e^{f(z)}$ is susceptible to the saddle point approximation, then $\dfrac{1}{2\pi i}\displaystyle\int_C F(z)dz \sim \frac{F(\zeta)}{\sqrt{2\pi f''(\zeta)}}$

a *general technique* for contour integration (not just for asymptotics)

**Proof.**

[ Similar to proof for SP bound; see text ]

# Saddle-point transfer theorem

**Theorem.** If a contour integral $\int_C F(z)dz$ with $F(z) = e^{f(z)}$ is susceptible to the saddle point approximation, then $\frac{1}{2\pi i}\int_C F(z)dz \sim \frac{F(z)}{\sqrt{2\pi f''(\zeta)}}$

**Saddle-point transfer.** Given a GF $G(z)$, if the contour integral of $G(z)/z^{N+1}$ along a path $C$ is susceptible to the saddle point approximation, then

$$\boxed{[z^N]G(z) = \frac{1}{2\pi i}\int_C G(z)\frac{dz}{z^{N+1}} \sim \frac{e^{g(\zeta)}}{\sqrt{2\pi g''(\zeta)}}}$$

where $g(z) = \ln G(z) - (N+1)\ln z$ and $\zeta$ is the unique positive real root of the *saddle point equation* $g'(z) = 0$.

Equivalent forms

SP equation

$$\frac{G'(z)}{G(z)} = \frac{N+1}{z}$$

SP approximation

$$\frac{G(\zeta)}{\zeta^{N+1}\sqrt{2\pi g''(\zeta)}}$$

**Proof.** Take $F(z) = G(z)/z^{N+1}$.

## Saddle point transfer example I: factorial/exponential

Goal. Estimate $\dfrac{1}{N!} = [z^N]e^z$

$$f(z) = \ln G(z) - (N+1)\ln z$$
$$= z - (N+1)\ln z$$
$$f'(z) = 1 - \frac{N+1}{z}$$
$$f''(z) = \frac{N+1}{z^2}$$

$$G(z) = e^z$$

Saddle point equation

$$f'(z) = 0$$

Saddle point

$$\zeta = N+1$$

Saddle point approx

$$[z^N]G(z) \sim \frac{G(\zeta)}{\zeta^{N+1}\sqrt{2\pi f''(\zeta)}}$$

Saddle point approximation

$$[z^N]e^z = \frac{1}{N!} \sim \frac{e^{N+1}}{(N+1)^{N+1}\sqrt{2\pi/(N+1)}}$$
$$\sim \frac{e^N}{N^N\sqrt{2\pi N}} \quad ✓$$

$$(1+\frac{1}{N})^N \to e$$

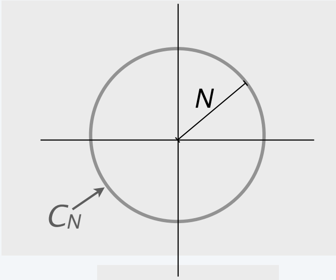
*Important note*: Need to check susceptibility, or use bound and sacrifice $\sqrt{2\pi N}$ factor.

tails are negligible, a central approximation holds, and tails can be completed back

## Saddle point method example I (susceptibility to saddle point)

Contour integral

$$\frac{1}{N!} = [z^N]e^z = \frac{1}{2\pi i}\int_{C_N} e^z \frac{dz}{z^{N+1}} = \frac{1}{2\pi i}\int_{C_N} e^{z-(N+1)\ln z}dz$$




$z = Ne^{i\theta}$

Switch to polar coordinates

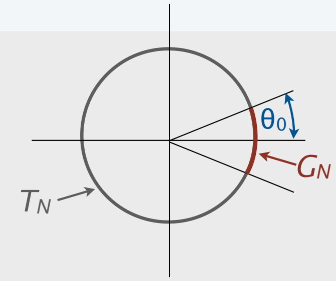
$$= \left(\frac{e}{N}\right)^N \frac{1}{2\pi}\int_0^{2\pi} e^{N(e^{i\theta}-1-i\theta)}d\theta$$

Split into central and tail contours

$$= \frac{1}{2\pi}\left(\frac{e}{N}\right)^N (Q_N + T_N)$$

$$Q_N = \int_{-\theta_0}^{+\theta_0} e^{N(e^{i\theta}-1-i\theta)}d\theta$$

$$T_N = \int_{\theta_0}^{2\pi-\theta_0} e^{N(e^{i\theta}-1-i\theta)}d\theta$$




exponentially small for
$\theta_0 = N^\alpha$ with $\alpha > -1/2$ [see text]

Neglect tails

$$\frac{1}{N!} \sim \frac{1}{2\pi}\frac{e^N}{N^N}Q_N$$

Note: Slightly shifting saddle point (from $N$+1 to $N$) simplifies calculations.

## Saddle point method example I (susceptibility to saddle point)

$$G_N = \int_{-\theta_0}^{+\theta_0} e^{N(e^{i\theta}-1-i\theta)}d\theta$$

Approximate integrand $= \int_{-\theta_0}^{+\theta_0} e^{-N\theta^2/2}d\theta\big(1 + O(N\theta_0^3)\big)$ 

$(e^{i\theta} - 1 - i\theta) = -\theta^2/2 + O(\theta^3)$

Restrict θ₀ to drop *O*-term $\sim \int_{-\theta_0}^{+\theta_0} e^{-N\theta^2/2}d\theta$ for $\theta_0 = N^\alpha$ with $\alpha < -1/3$

Change of variable $\sim \frac{1}{\sqrt{N}}\int_{-\theta_0\sqrt{N}}^{+\theta_0\sqrt{N}} e^{-t^2/2}dt$

$\theta = t/\sqrt{N}$

$d\theta = dt/\sqrt{N}$

Restrict θ₀ to complete tails $\sim \frac{1}{\sqrt{N}}\int_{-\infty}^{+\infty} e^{-t^2/2}dt$ for $\theta_0 = N^\alpha$ with $\alpha > -1/2$

Collect restrictions $\sim \sqrt{2\pi/N}$ for $\theta_0 = N^{2/5}$

$\int_{N^{1/2-\alpha}}^{\infty} e^{-t^2/2}dt = O(e^{-N^{1-2\alpha}})$

Finish $\frac{1}{N!} \sim \frac{1}{2\pi}\left(\frac{e}{N}\right)^N G_N = \left(\frac{e}{N}\right)^N \frac{1}{\sqrt{2\pi N}}$ ✓

## Saddle-point asymptotics

Q. $N^{1/2-2/5} = N^{1/10}$. Aren't we touching on N needing to be in the "galactic" range?

not relevant in this galaxy

A. Those estimates are in the *exponent*.
Ex. $e^{-N^{1/10}} = e^{-8} \doteq .000335$ when $N$ is $2^{30}$ (about 1 billion).

A. Methods extend to derive full asymptotic series to any desired precision.

A. Results are easy to validate numerically.

A. Towards goal of general schema cover whole families of combinatorial classes.

## Saddle point transfer example II: Catalan/central binomial

**Goal.** Estimate $\binom{2N}{N} = [z^N](1+z)^{2N}$

$$G(z) = (1+z)^{2N}$$

$$f(z) = \ln G(z) - (N+1)\ln z$$
$$= 2N\ln(1+z) - (N+1)\ln z$$
$$f'(z) = \frac{2N}{1+z} - \frac{N+1}{z}$$
$$f''(z) = -\frac{2N}{(1+z)^2} + \frac{N+1}{z^2}$$

Saddle point equation

$$f'(z) = 0$$

Saddle point $$\zeta = \frac{N+1}{N-1}$$

Saddle point approx

$$[z^N]G(z) \sim \frac{G(\zeta)}{\zeta^{N+1}\sqrt{2\pi f''(\zeta)}}$$

Saddle point approximation $$[z^N](1+z)^{2N} = \binom{2N}{N} \sim \frac{\left(\frac{2N}{N-1}\right)^{2N}}{\left(\frac{N+1}{N-1}\right)^{N+1}\sqrt{2\pi f''\left(\frac{N+1}{N-1}\right)}}$$

*Note*: Slight shift of saddle point often simplifies calcuations (see next slide).

## Saddle point transfer example II: Catalan/central binomial

**Goal.** Estimate $\binom{2N}{N} = [z^N](1+z)^{2N}$

$$G(z) = (1+z)^{2N}$$

$$f(z) = \ln G(z) - (N+1)\ln z$$
$$= 2N\ln(1+z) - (N+1)\ln z$$
$$f'(z) = \frac{2N}{1+z} - \frac{N+1}{z}$$
$$f''(z) = -\frac{2N}{(1+z)^2} + \frac{N+1}{z^2}$$

Saddle point equation

$$f'(z) = 0$$

Saddle point

$$\zeta = \frac{N+1}{N-1} \sim 1$$

Saddle point approx

$$[z^N]G(z) \sim \frac{G(\zeta)}{\zeta^{N+1}\sqrt{2\pi f''(\zeta)}}$$

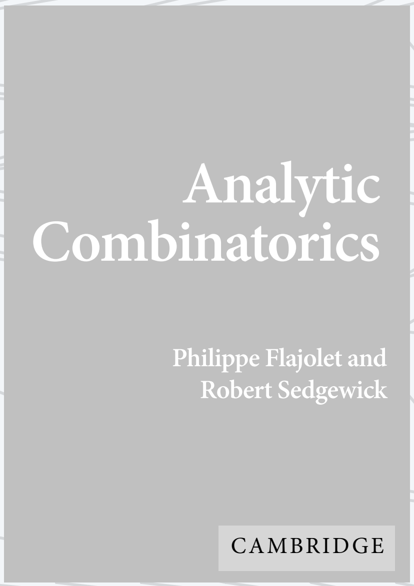
$$f''(1) \sim N/2$$

Saddle point approximation

$$[z^N](1+z)^{2N} = \binom{2N}{N} \sim \frac{4^N}{\sqrt{\pi N}}$$

*Important note*: Need to check susceptibility, or use bound and sacrifice $\sqrt{\pi N}$ factor.

tails are negligible, a central approximation holds, and tails can be completed back

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# 8. Saddle-Point Asymptotics

- Modulus surfaces
- Saddle point bounds
- **Saddle point asymptotics**
- Applications

Analytic Combinatorics

Philippe Flajolet and Robert Sedgewick

CAMBRIDGE

http://ac.cs.princeton.edu

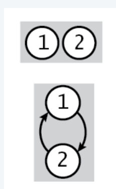
# 8. Saddle-Point Asymptotics

- Modulus surfaces
- Saddle point bounds
- Saddle point asymptotics
- **Applications**

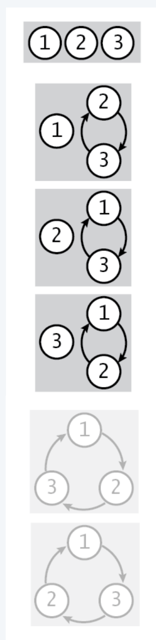
# Involutions

Q. How many different permutations of size $N$ with no cycle lengths >2 ?

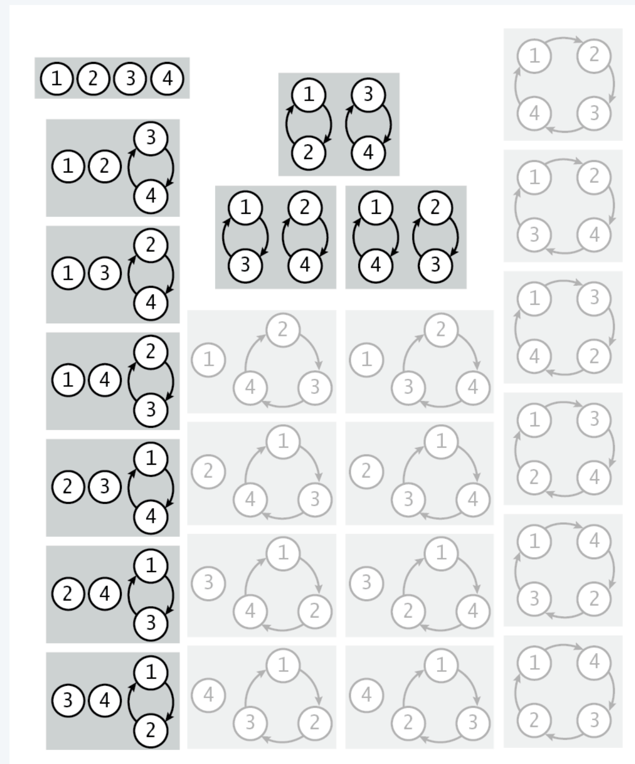
$I_1 = 1$



$I_2 = 2$

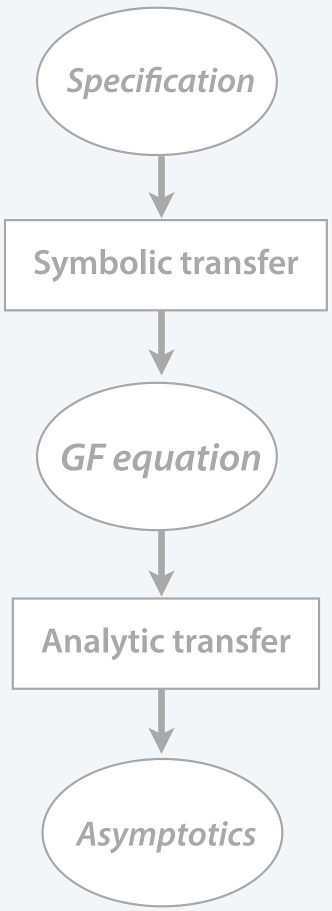


$I_3 = 4$



$I_4 = 10$

# AC example with saddle-point asymptotics: Involutions



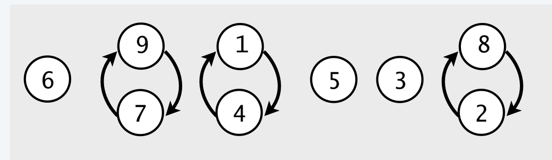

**I**, the class of involutions

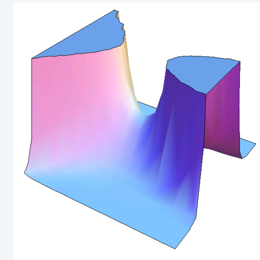
**I** = SET(CYC$_{1,2}$(**Z**))

$$I(z) = e^{z+z^2/2}$$

$$[z^N]I(z) \sim \frac{e^{N/2+\sqrt{N}-1/4}}{2N^{N/2}\sqrt{\pi N}}$$

$$N![z^N]I(z) \sim \frac{1}{\sqrt[4]{4e}}\left(\frac{N}{e}\right)^{N/2} e^{\sqrt{N}}$$



**Saddle-point transfer.** Given a GF $G(z)$, if the contour integral $\frac{1}{2\pi i}\int_C G(z)\frac{dz}{z^{N+1}}$ is susceptible to the saddle point approximation, then

$$\boxed{[z^N]G(z) = \frac{1}{2\pi i}\int_C G(z)\frac{dz}{z^{N+1}} \sim \frac{e^{g(\zeta)}}{\sqrt{2\pi g''(\zeta)}}}$$

where $g(z) = \ln G(z) - (N+1)\ln z$ and $\zeta$ is the unique positive real root of the *saddle point equation* $g'(z) = 0$ (equivalently, $G'(z)/G(z) = (N+1)/z$ ).

$$g(z) = z + z^2/2 - (N+1)\ln z$$

$$g'(z) = 1 + z - \frac{N+1}{z}$$

$$g''(z) = 1 + \frac{N+1}{z^2}$$

$$\zeta^2 + \zeta - (N+1) = 0$$

$$\zeta = -\frac{1}{2} + \frac{1}{2}\sqrt{1+4(N+1)}$$

$$\sim \sqrt{N} - 1/2 + O(1/\sqrt{N})$$

Important note: *Need to check susceptibility.*

- generally more difficult than for other transfer thms.
- option: use bound (sacrifice $\sqrt{2\pi N}$ factor.

## Set partitions

Q. How many ways to *partition* a set of size of $N$ ?

| |
|---|
| {1} |
| $S_1 = 1$ |

| |
|---|
| {1} {2} |
| {1 2} |
| $S_2 = 2$ |

| |
|---|
| {1} {2} {3} |
| {1} {2 3} |
| {2} {1 3} |
| {3} {1 2} |
| {1} {2} {3} |
| $S_3 = 5$ |

| |
|---|
| {1} {2} {3} {4} |
| {1} {2 3 4} |
| {2} {1 3 4} |
| {3} {1 2 4} |
| {4} {1 2 3} |
| {1 2} {3} {4} |
| {1 3} {2} {4} |
| {1 4} {2} {3} |
| {2 3} {1} {4} |
| {2 4} {1} {3} |
| {3 4} {1} {2} |
| {1 2} {3 4} |
| {1 3} {2 4} |
| {1 4} {2 3} |
| {1 2 3 4} |
| $S_4 = 15$ |

# AC example with saddle-point asymptotics: Set partitions

*Specification*

↓

Symbolic transfer

↓

*GF equation*

↓

Analytic transfer

↓

*Asymptotics*

**S**, the class of set partitions

$$\mathbf{S} = \mathrm{SET}(\mathrm{SET}_{>0}(\mathbf{Z}))$$

↓

$$S(z) = e^{e^z - 1}$$

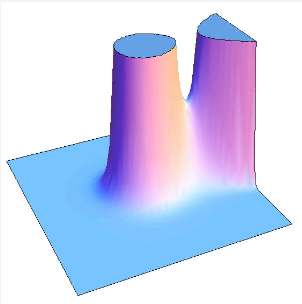
↓

[complex expression: use bound]

$$S_N \leq N! \frac{e^{N-1}}{(\ln N)^N} \sim \left(\frac{N}{\ln N}\right)^N \sqrt{2\pi N/e}$$

```
{2 3} {5 7 9} {4} {1 8}
```

$e^{e^z-1}/z^6$



Saddle-point transfer. Given a GF $G(z)$, if the contour integral $\frac{1}{2\pi i}\int_C G(z)\frac{dz}{z^{N+1}}$ is susceptible to the saddle point approximation, then

$$\boxed{[z^N]G(z) = \frac{1}{2\pi i}\int_C G(z)\frac{dz}{z^{N+1}} \sim \frac{e^{g(\zeta)}}{\sqrt{2\pi g''(\zeta)}}}$$

where $g(z) = \ln G(z) - (N+1)\ln z$ and $\zeta$ is the unique positive real root of the *saddle point equation* $g'(z) = 0$ (equivalently, $G'(z)/G(z) = (N+1)/z$ ).

$$g(z) = e^z - 1 - (N+1)\ln z$$

$$g'(z) = e^z - \frac{N+1}{z}$$

$$g''(z) = e^z + \frac{N+1}{z^2}$$

$$\zeta e^\zeta = N+1$$

$$\zeta \sim \ln N - \ln\ln N$$

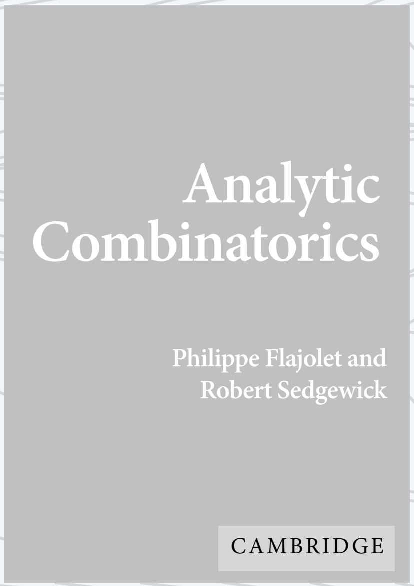
SP bound

$$[z^N]S(z) \leq \frac{G(\zeta)}{\zeta^N}$$

## Saddle point: summary of combinatorial applications

| | construction | GF | saddle point bound | coefficient asymptotics |
|---|---|---|---|---|
| urns | $\mathbf{U} = SET(\mathbf{Z})$ | $e^z$ | $\to \dfrac{e^N}{N^N}$ | $\dfrac{1}{N!} \sim \dfrac{e^N}{N^N\sqrt{2\pi N}}$ |
| central binomial | | $[z^N](1+z)^{2N}$ | $\to 4^N$ | $\sim \dfrac{4^N}{\sqrt{\pi N}}$ |
| involutions | $\mathbf{I} = SET(CYC_{1,2}(\mathbf{Z}))$ | $e^{z+z^2/2}$ | $\le N!\dfrac{e^{N/2+\sqrt{N}-1/4}}{\sqrt{2}N^{N/2}}$ | $\sim N!\dfrac{e^{N/2+\sqrt{N}-1/4}}{2N^{N/2}\sqrt{\pi N}}$ |
| set partitions | $\mathbf{S} = SET(SET_{>0}(\mathbf{Z}))$ | $e^{e^z-1}$ | $\le N!\dfrac{e^{N-1}}{(\ln N)^N}$ | |
| fragmented permutations | $\mathbf{F} = SET(SEQ_{>0}(\mathbf{Z}))$ | $e^{z/(1-z)}$ | $\le N! e^{2\sqrt{N}-1/2}$ | $\le N!\dfrac{e^{2\sqrt{N}-1/2}}{2\sqrt{\pi}N^{3/4}}$ |
| integer partitions | $\mathbf{P} = MSET(SEQ_{>0}(\mathbf{Z}))$ | $e^{z/(1-z)+z^2/2(1-z^2)+\ldots}$ | $\le e^{\pi\sqrt{2N/3}}$ | $\sim \dfrac{e^{\pi\sqrt{2N/3}}}{4N\sqrt{3}}$ |

*not for amateurs*

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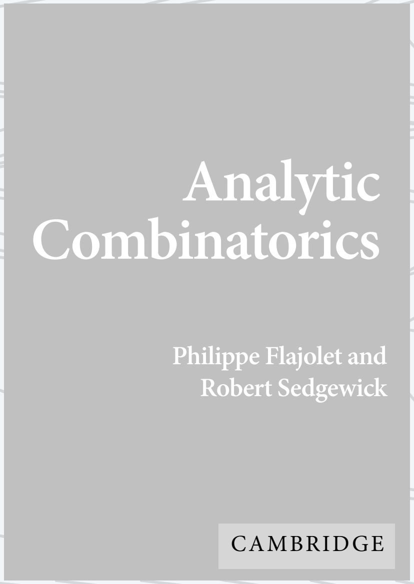
# 8. Saddle-Point Asymptotics

- Modulus surfaces
- Saddle point bounds
- Saddle point asymptotics
- **Applications**

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# 8. Saddle-Point Asymptotics

- Modulus surfaces
- Saddle point bounds
- Saddle point asymptotics
- Applications
- **AC wrapup**

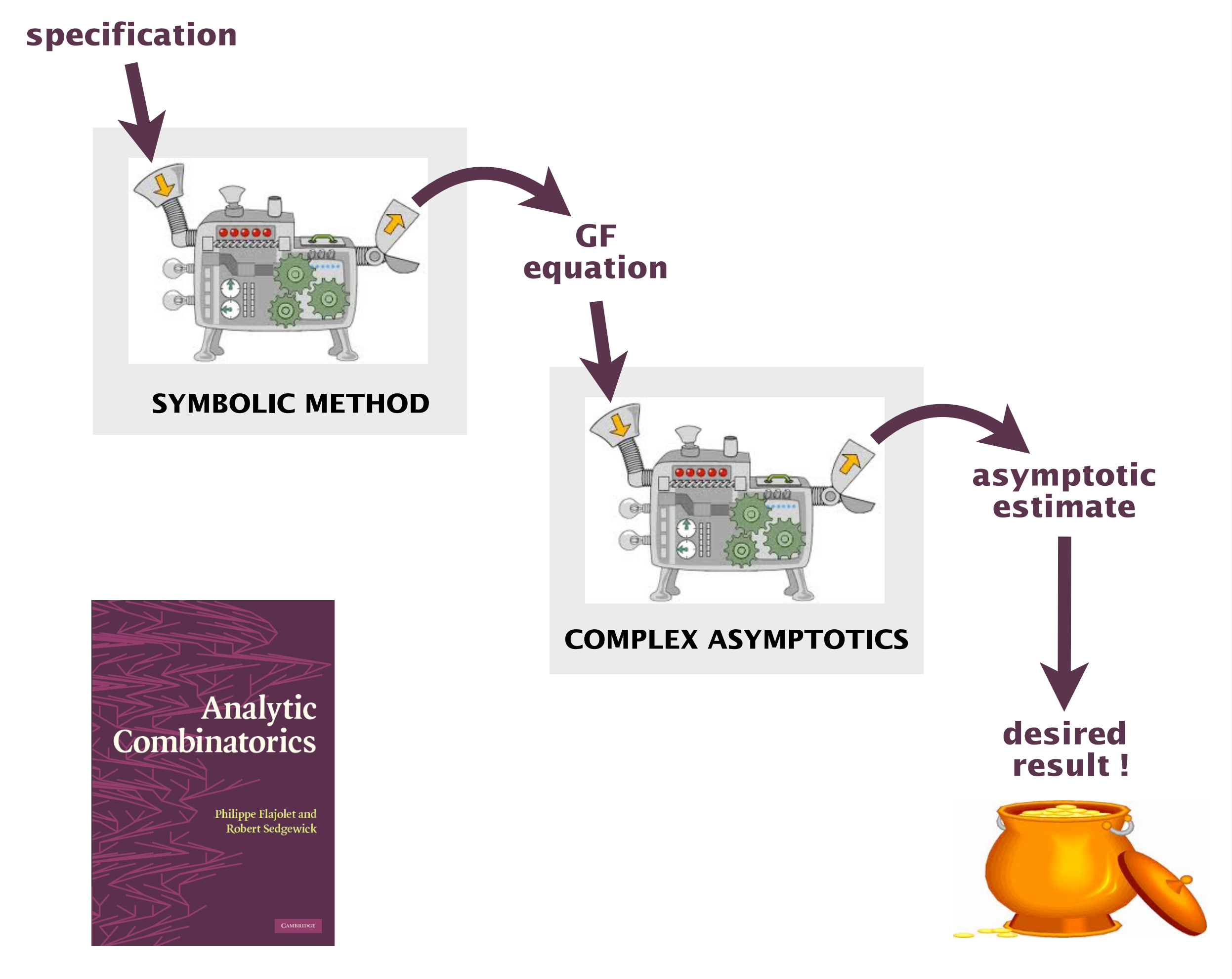
# Analytic combinatorics overview

**A. SYMBOLIC METHOD**

1. OGFs
2. EGFs
3. MGFs

**B. COMPLEX ASYMPTOTICS**

4. Rational & Meromorphic
5. Applications of R&M
6. Singularity Analysis
7. Applications of SA
8. Saddle point



## Basic ideas of analytic combinatorics (summary)

**1.** *Combinatorial specifications* provide succinct definitions of a wide range of discrete structures.

**2.** The *symbolic method* transforms specifications to equations that define *generating functions*.

**3.** *Complexification* treats generating functions as *analytic* objects, giving estimates of coefficients.

*Cauchy's coefficient formula* gives coefficient asymptotics when singularities are poles.

*Singularity analysis* provides a general approach to analyzing GFs with essential singularities.

*Saddle-point asymptotics* is effective for functions with no singularities.

**4.** Combinatorial classes fall into general *schema* that are governed by *universal* asymptotic laws.

# Constructions and symbolic transfers

### The symbolic method for unlabeled objects (summary)

| operation | notation | semantics | OGF |
|---|---|---|---|
| *disjoint union* | $A + B$ | disjoint copies of objects from $A$ and $B$ | $A(z) + B(z)$ |
| *Cartesian product* | $A \times B$ | ordered pairs of copies of objects, one from $A$ and one from $B$ | $A(z)B(z)$ |
| *sequence* | $SEQ(A)$ | sequences of objects from $A$ | 1 |
| *powerset* | $PSET(A)$ | finite sets of objects from $A$ (no repetitions) | $\prod_{n\geq1}(1+z$ |
| *multiset* | $MSET(A)$ | finite sets of objects from $A$ (with repetitions) | $\prod_{n\geq1}$ (1 |

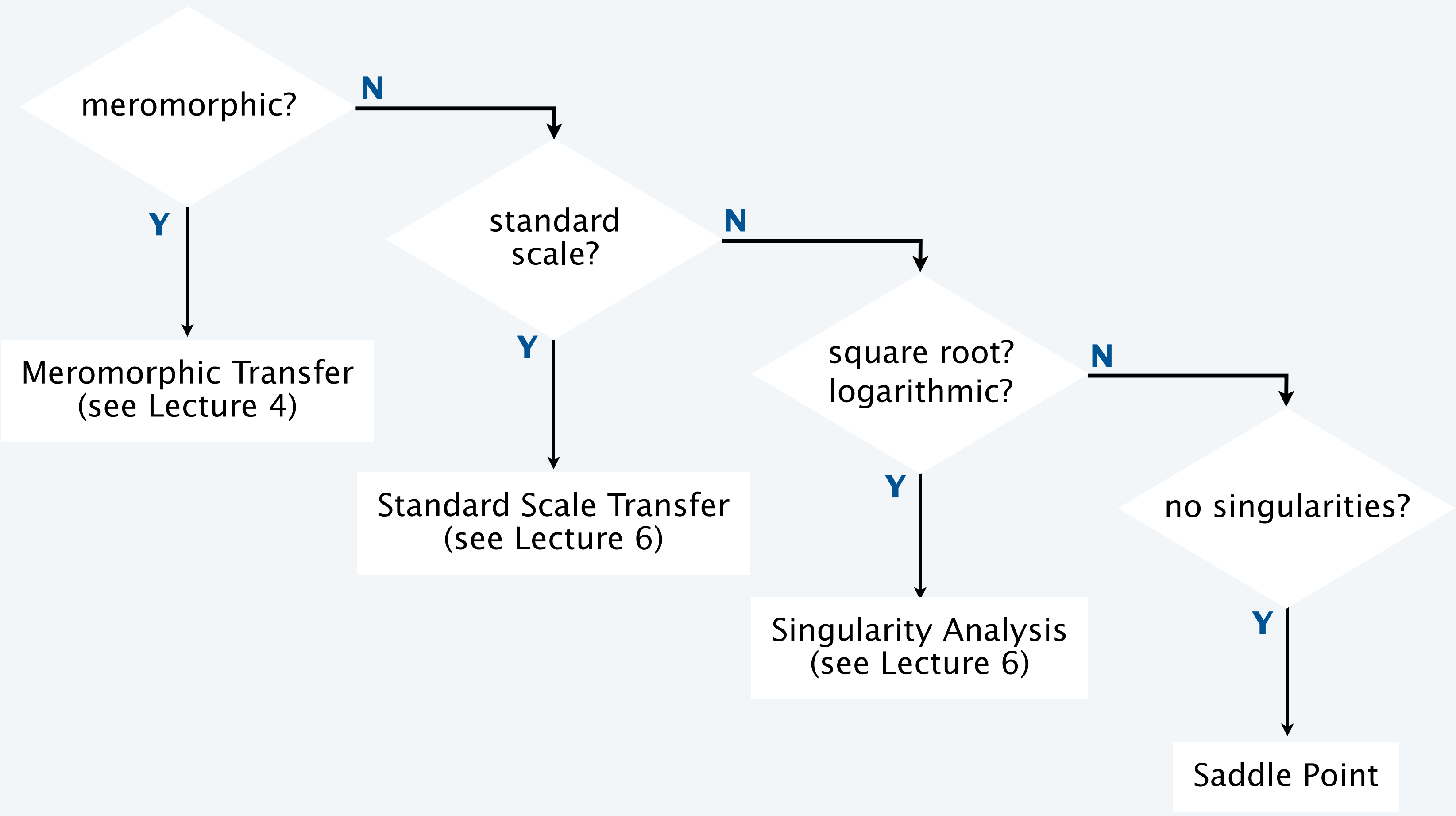
Additional constructs are available (and still being invented)—one exa

### The symbolic method for labelled classes (transfer theorem)

Theorem. Let $A$ and $B$ be combinatorial classes of labelled objects with EGFs $A(z)$ and $B(z)$. Then

| *construction* | *notation* | *semantics* | *EGF* |
|---|---|---|---|
| disjoint union | $A + B$ | disjoint copies of objects from $A$ and $B$ | $A(z) + B(z)$ |
| labelled product | $A \star B$ | ordered pairs of copies of objects, one from $A$ and one from $B$ | $A(z)B(z)$ |
| sequence | $SEQ_k(A)$ or $A^k$ | $k$- sequences of objects from $A$ | $A(z)^k$ |
| | $SEQ(A)$ | sequences of objects from $A$ | $\dfrac{1}{1-A(z)}$ |
| set | $SET_k(A)$ | $k$-sets of objects from $A$ | $A(z)^k/k!$ |
| | $SET(A)$ | sets of objects from $A$ | $e^{A(z)}$ |
| cycle | $CYC_k(A)$ | $k$-cycles of objects from $A$ | $A(z)^k/k$ |
| | $CYC(A)$ | cycles of objects from $A$ | $\ln\dfrac{1}{1-A(z)}$ |

# Explicit analytic transfers



# Schemas

Combinatorial problems can be organized into broad schemas, covering infinitely many combinatorial types and governed by simple asymptotic laws.

Theorem. *Asymptotics of exp-log labelled sets.*

Suppose that a labelled set class **F** = SET$_\Phi$(**G**) is exp-log($\alpha$, $\beta$, $\rho$) with $G(z) \sim \alpha \log \frac{1}{1 - z/\rho} + \beta$. Then $F(z) \sim e^\beta \big(\frac{1}{1 - z/\rho}\big)^\alpha$ and $[z^N]F(z) \sim \frac{e^\beta}{\Gamma(\alpha)} \big(\frac{1}{\rho}\big)^N N^{1-\alpha}$

Theorem. *Asymptotics of supercritical sequences.* If **F** = SEQ(**G**) is a strongly aperiodic supercritical sequence class, then $[z^N]F(z) \sim \frac{1}{G'(\lambda)} \frac{1}{\lambda^{N+1}}$ where $\lambda$ is the root of $G(\lambda) = 1$ in (0, $\rho$).

radius of convergence of $G(z)$

Theorem. If C is an irreducible context-free class, then its generating function $C(z)$ has a square-root singularity at its radius of convergence $\rho$. If $C(z)$ is aperiodic, then the dominant singularity is unique and $[z^N]F(z) \sim \frac{1}{\sqrt{\alpha\pi}} \big(\frac{1}{\rho}\big)^N N^{-3/2}$ where $\alpha$ is a computable real.

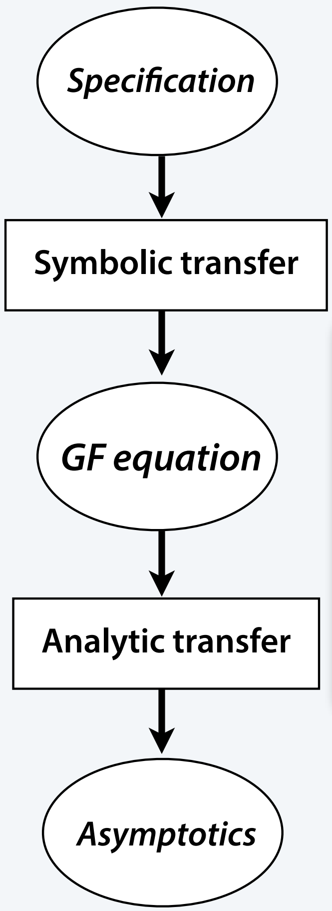
Theorem. If a simple variety of trees with GF $F(z) = z\phi(F(z))$ is $\lambda$-invertible (where $\lambda$ is the positive real root of $\phi(u) = u\phi'(u)$ ) then $[z^N]F(z) \sim \frac{1}{\sqrt{2\pi\phi''(\lambda)/\phi(\lambda)}} \big(\phi'(\lambda)\big)^N N^{-3/2}$

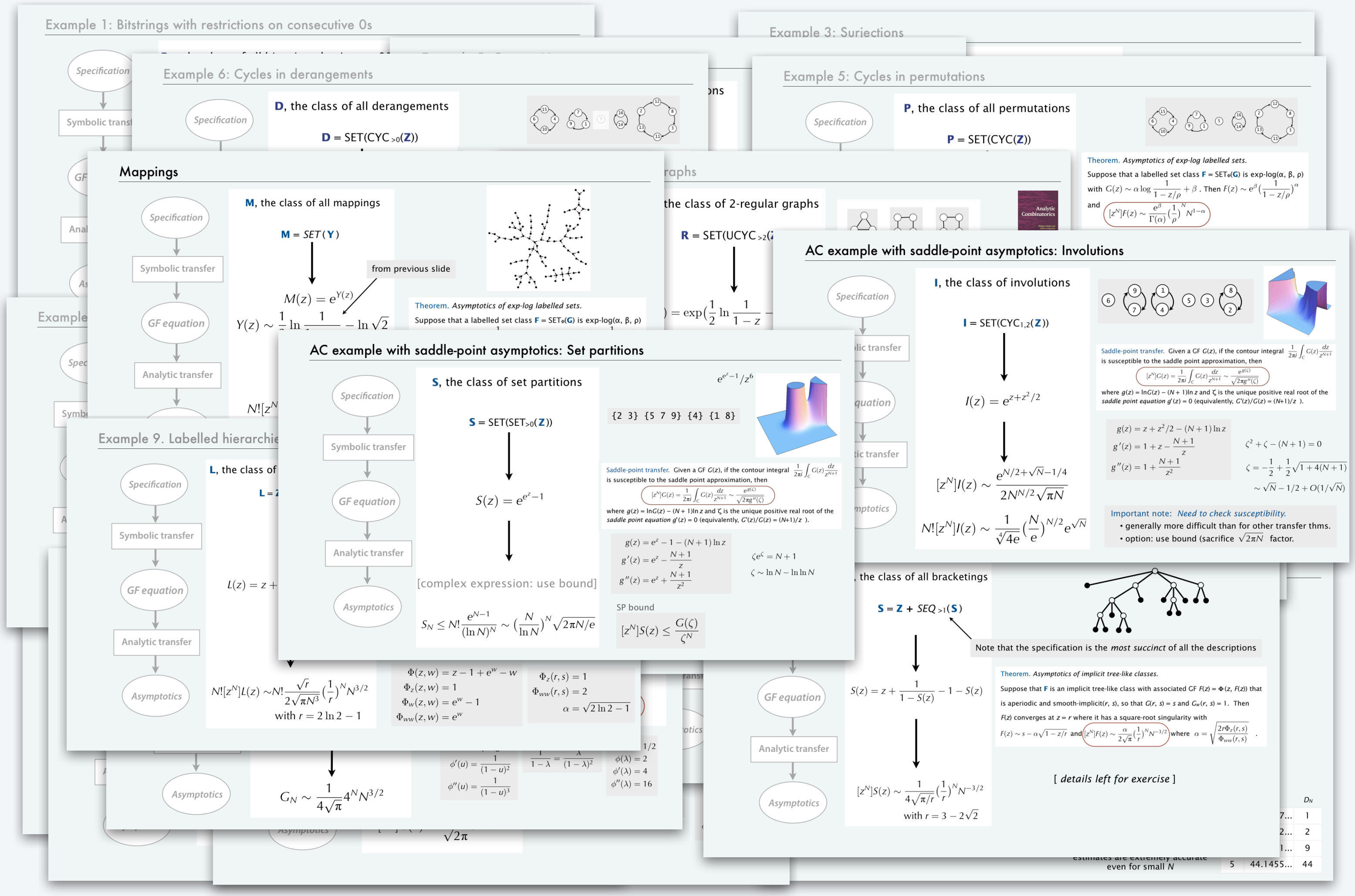
Theorem. *Asymptotics of implicit tree-like classes.*

Suppose that **F** is an implicit tree-like class with associated GF $F(z) = \Phi(z, F(z))$ that is aperiodic and smooth-implicit($r$, $s$), so that $G(r, s) = s$ and $G_w(r, s) = 1$. Then $F(z)$ converges at $z = r$ where it has a square-root singularity with $F(z) \sim s - \alpha\sqrt{1 - z/r}$ and $[z^N]F(z) \sim \frac{\alpha}{2\sqrt{\pi}} \big(\frac{1}{r}\big)^N N^{-3/2}$ where $\alpha = \sqrt{\frac{2r\Phi_z(r, s)}{\Phi_{ww}(r, s)}}$ .

The discovery of such schemas and of the associated universality properties constitues the *very essence* of analytic combinatorics.

# “If you can specify it, you can analyze it”



## What is "Analytic combinatorics"?

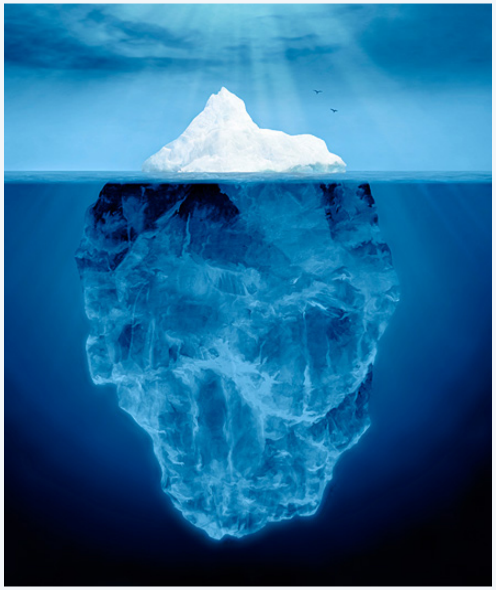
[ In case someone asks... ]

*Analytic combinatorics aims to enable precise quantitative predictions of the properties of large combinatorial structures. The theory has emerged over recent decades as essential both for the analysis of algorithms and for the study of scientific models in other disciplines, including statistical physics, computational biology, and information theory.*

# What’s next?

Suggestions for further study in *Analytic Combinatorics*

- Additional constructions and associated symbolic transfers
- Applications to paths in lattices and many other types
- Details of SA proofs
- Periodicity, irreducibility, algebraic functions
- Additional schema
- Drmota-Llaley-Woods theorem
- Technical conditions for SP approximations
- Multivariate asymptotics and limit laws
- Applications, applications, applications, *applications*



Available as "postscript" to this course

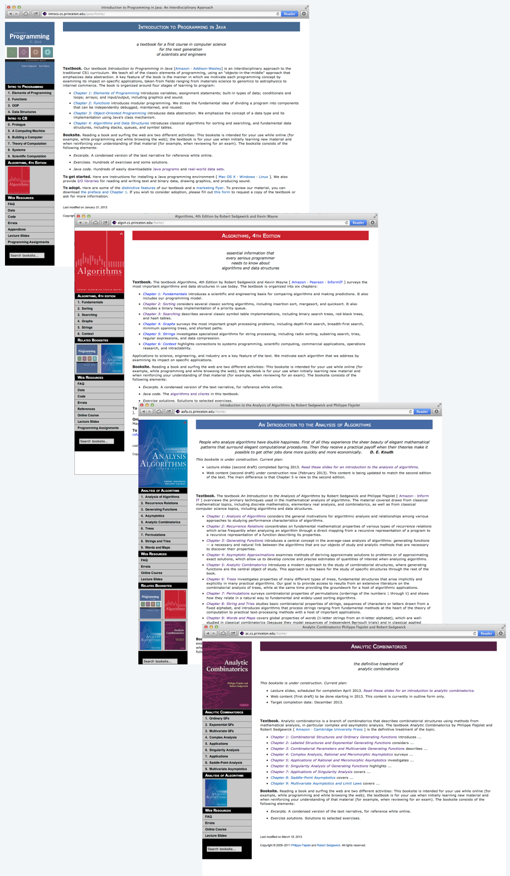
For an overview of Flajolet's work and current research in AC, watch the lecture

*"If you can specify it you can analyze it": the lasting legacy of Philippe Flajolet*

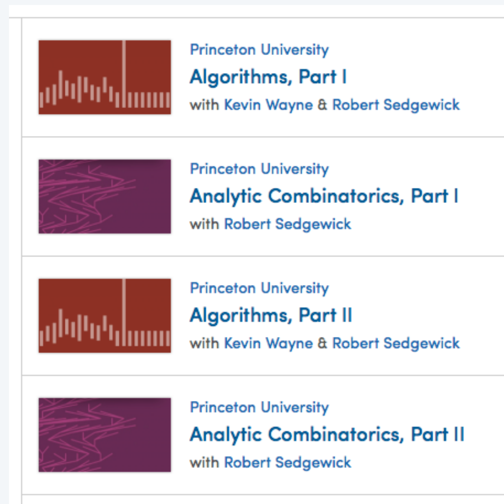
# Shameless plugs

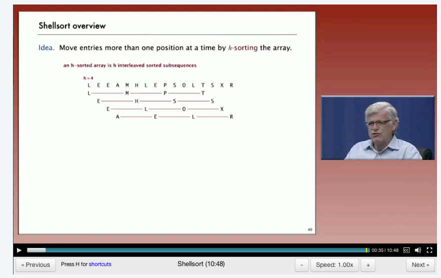
Books


Booksites



Online courses



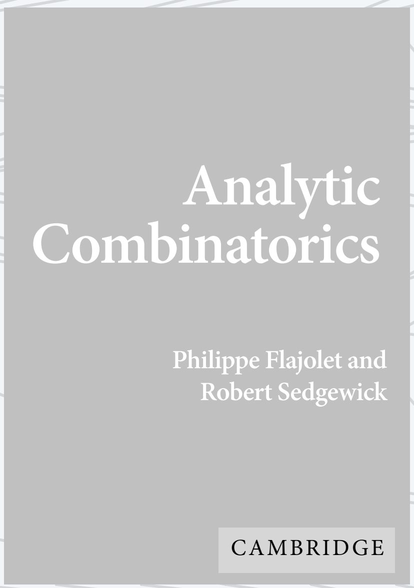

And, especially for students in *this* course . . .

a T-shirt!

see AC booksite
for details

*Now this is not the end. It is not even the beginning of the end.*
*But it is, perhaps, the end of the beginning. ”*

*— Winston Churchill, 1942*



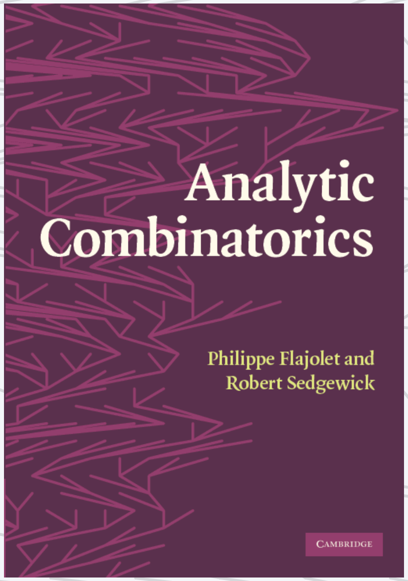

http://ac.cs.princeton.edu

# 8. Saddle-Point Asymptotics

- Modulus surfaces
- Saddle point bounds
- Saddle point asymptotics
- Applications
- **AC wrapup**

ANALYTIC COMBINATORICS

PART TWO

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# 8. Saddle-Point Asymptotics