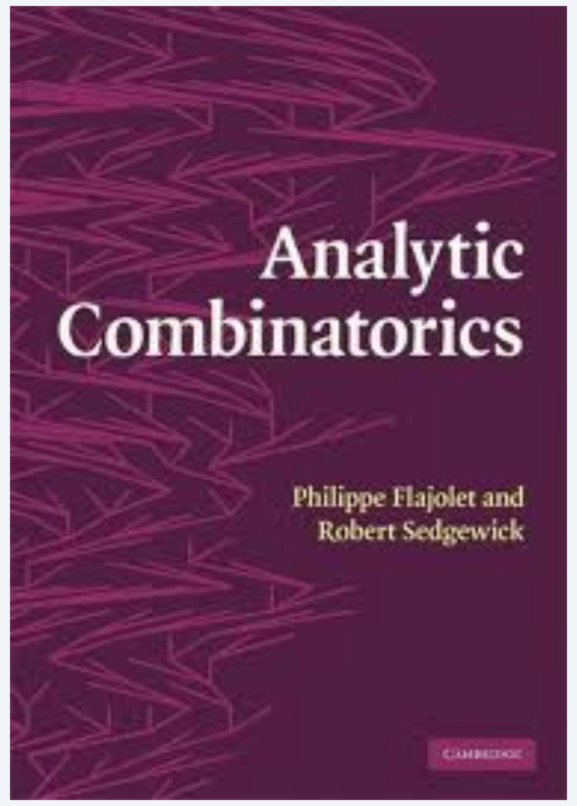
Plan for second half of the course

Lectures from Analytic Combinatorics

- One or two lectures, posted weekly.
- Emphasis on lectures, with reference to the book.



More creative exercises

- May require code.
- (Slightly) more emphasis on Q&As.

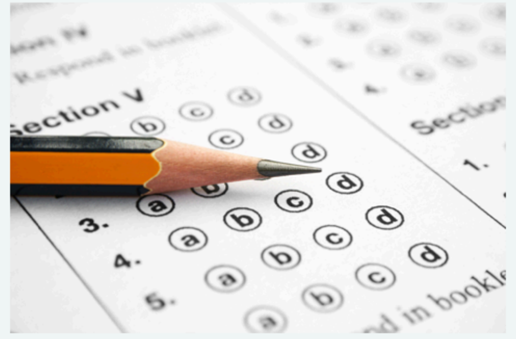
“Questions and Answers” (Q&As)

One topic of class meetings for COS 488 is to develop good questions for future exams.

Properties of a good exam question.

- Easy to understand.
- Easy to grade.
- Solvable in 10 minutes or less (but not trivial).
- Tests understanding of an important topic.
- “Fair” (no tricks)
- Teaches something (optional but desirable)

Your grade will be based on these criteria!



For examples, see Q&A from *Analysis of Algorithms* (selected examples to follow).

AC basics Q&A 1

Q. Match each description on the left to 0 or 1 of the symbolic specifications on the right (*ODD* denotes the set of odd numbers).

$\mathrm{SEQ}_{ODD}(\mathrm{SEQ}(Z))$

$\mathrm{SEQ}(\mathrm{SEQ}_{ODD}(Z))$

$\mathrm{SEQ}(Z)$

$\mathrm{SEQ}_{M}(Z)$

$\mathrm{MSET}(Z)$

$\mathrm{MSET}_{ODD}(Z)$

$\mathrm{PSET}(Z)$

$\mathrm{MSET}(\mathrm{SEQ}_{ODD}(Z))$

$\mathrm{MSET}_{ODD}(\mathrm{SEQ}(Z))$

compositions

compositions into *M* parts

compositions into odd parts

compositions into an odd number of parts

compositions into distinct parts

partitions

partitions into *M* parts

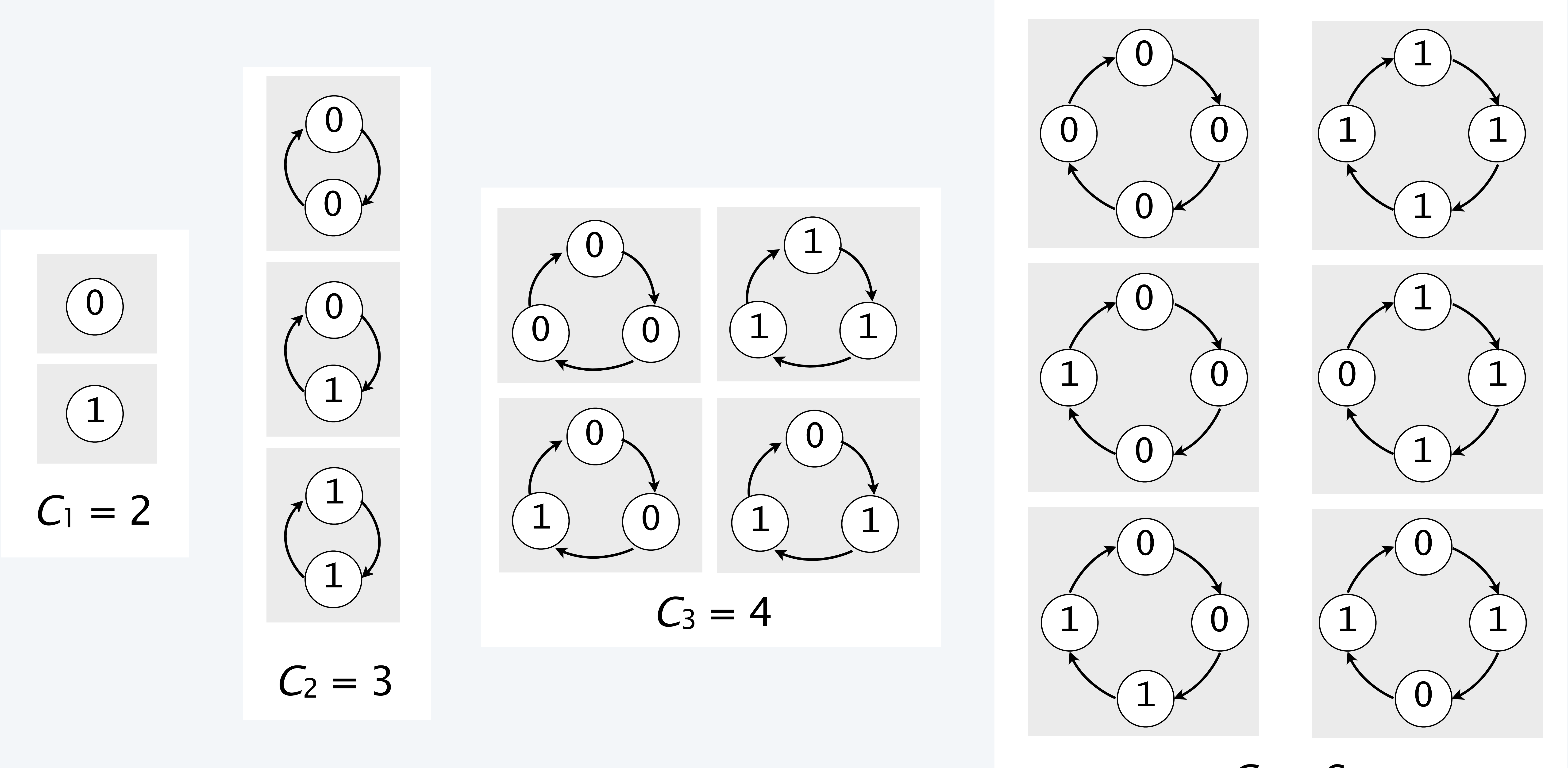
partitions into odd parts

partitions into an odd number of parts

partitions into distinct parts

AC Basics Q&A 2: cyclic bitstrings

Def. A *cyclic bitstring* is a cyclic sequence of bits



$C_1 = 2$

$C_2 = 3$

$C_3 = 4$

$C_4 = 6$

$C_5 = 8$

Q. How many N-bit cyclic bitstrings ?

Q&A example: cyclic bitstrings

Q. How many N-bit cyclic bitstrings ?

One possibility

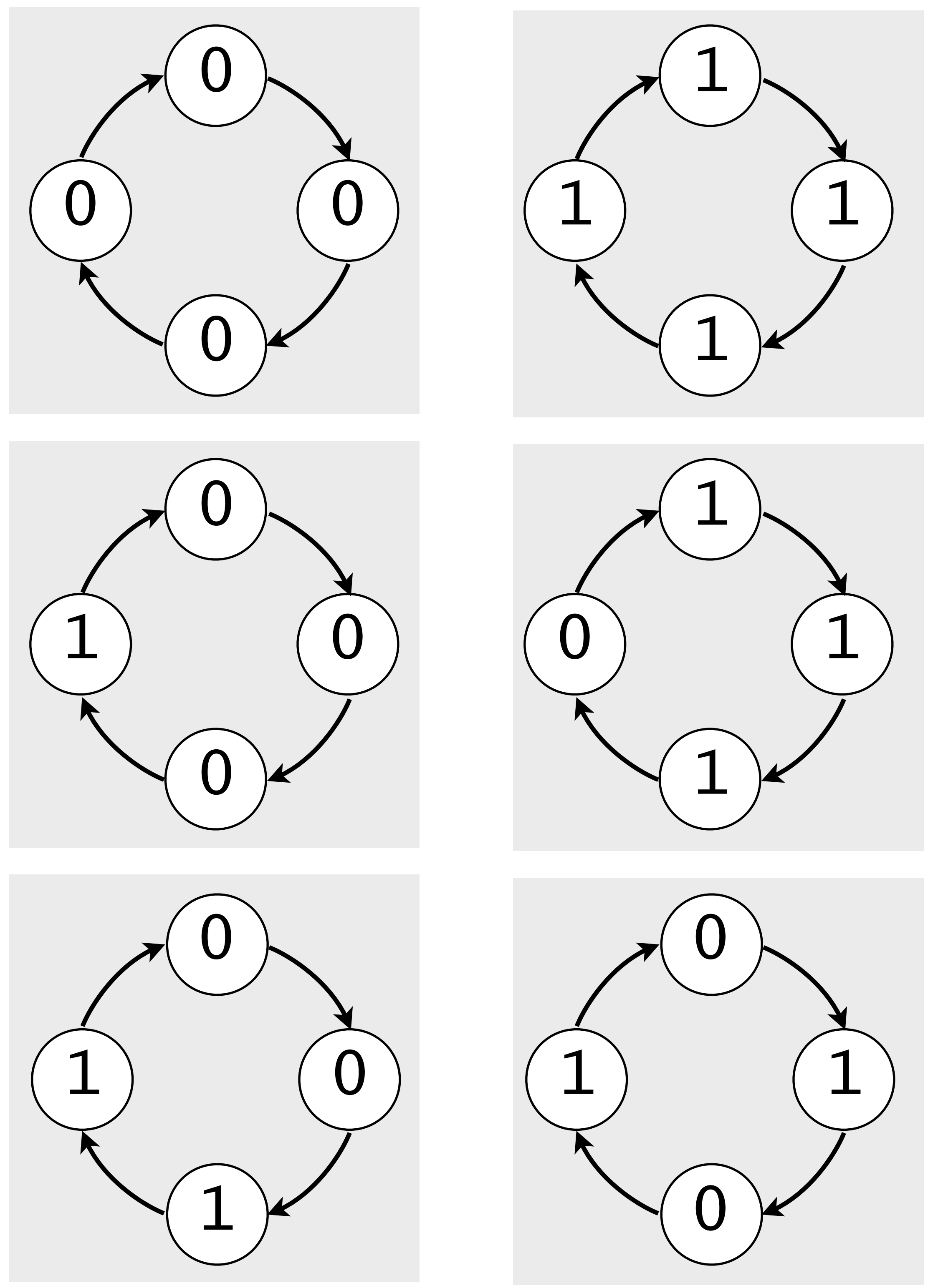
- Solution is “easy”.
- Create an exam question with appropriate hints.

Another possibility

- Solution is “difficult” or “complicated”.
- Figure out a way to simplify.
- Or, think about a different problem.

Third possibility

- Problem you thought of is a “classic”.
- Use OEIS.



$C_4 = 6$

THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES®

founded in 1964 by N. J. A. Sloane

2, 3, 4, 6, 8

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A000031 Number of n-bead necklaces with 2 colors when turning over is not allowed; also number of output sequences from a simple n-stage cycling shift register; also number of binary irreducible polynomials whose degree divides n. +20 83

(Formerly M0564 N0203)

1, **2**, **3**, **4**, **6**, **8**, 14, 20, 36, 60, 108, 188, 352, 632, 1182, 2192, 4116, 7712, 14602, 27596, 52488, 99880, 190746, 364724, 699252, 1342184, 2581428, 4971068, 9587580, 18512792, 35792568, 69273668, 134219796, 260301176, 505294128, 981706832 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,2

COMMENTS

Also a(n)-1 is the number of 1's in the truth table for the lexicographically least de Bruijn cycle (Fredricksen).

In music, a(n) is the number of distinct classes of scales and chords in an n-note equal-tempered tuning system. - Paul Cantrell, Dec 28 2011

REFERENCES

S. W. Golomb, Shift-Register Sequences, Holden-Day, San Francisco, 1967, pp. 120, 172.

R. M. May, Simple mathematical models with very complicated dynamics, Nature, 261 (Jun 10, 1976), 459-467.

N. J. A. Sloane, A Handbook of Integer Sequences, Academic Press, 1973 (includes this sequence).

N. J. A. Sloane and Simon Plouffe, The Encyclopedia of Integer Sequences, Academic Press, 1995 (includes this sequence).

R. P. Stanley, Enumerative Combinatorics, Cambridge, Vol. 2, 1999; see Problem 7.112(a).

LINKS

T. D. Noe and Seiichi Manyama, Table of n, a(n) for n = 0..3333 (first 201 terms from T. D. Noe)

Joerg Arndt, Matters Computational (The Fxtbook), p. 151, pp. 379-383.

P. J. Cameron, Sequences realized by oligomorphic permutation groups, J. Integ. Seqs. Vol. 3 (2000), #00.1.5.

S. N. Ethier and J. Lee, Parrondo games with spatial dependence, arXiv preprint arXiv:1202.2609 [math.PR], 2012. - From N. J. A. Sloane, Jun 10 2012

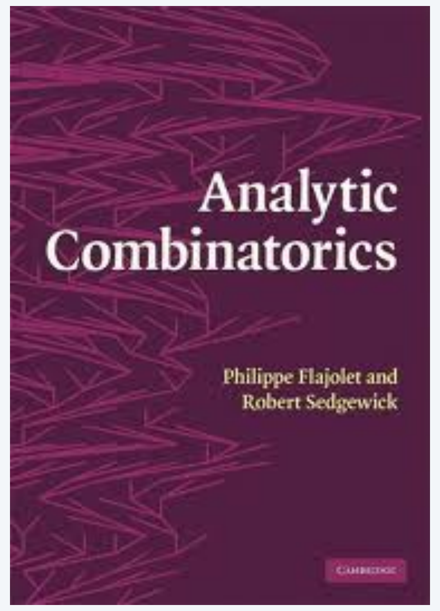
S. N. Ethier, Counting toroidal binary arrays, arXiv preprint arXiv:1301.2352 [math.CO], 2013.

N. J. Fine, Classes of periodic sequences, Illinois J. Math., 2 (1958), 285-302.

P. Flajolet and R. Sedgewick, Analytic Combinatorics, 2009; see pages 18, 64.

H. Fredricksen, The lexicographically least de Bruijn cycle, J. Combin. Theory, 9 (1970) 1-5.

Harold Fredricksen, An algorithm for generating necklaces of beads in two colors, Discrete Mathematics, Volume 61, Issues 2-3, September 1986, Pages 181-188.



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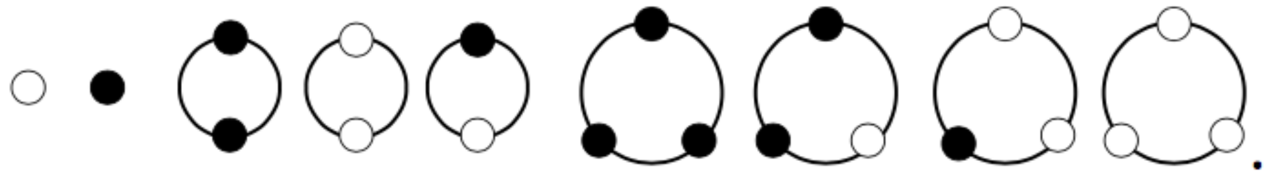
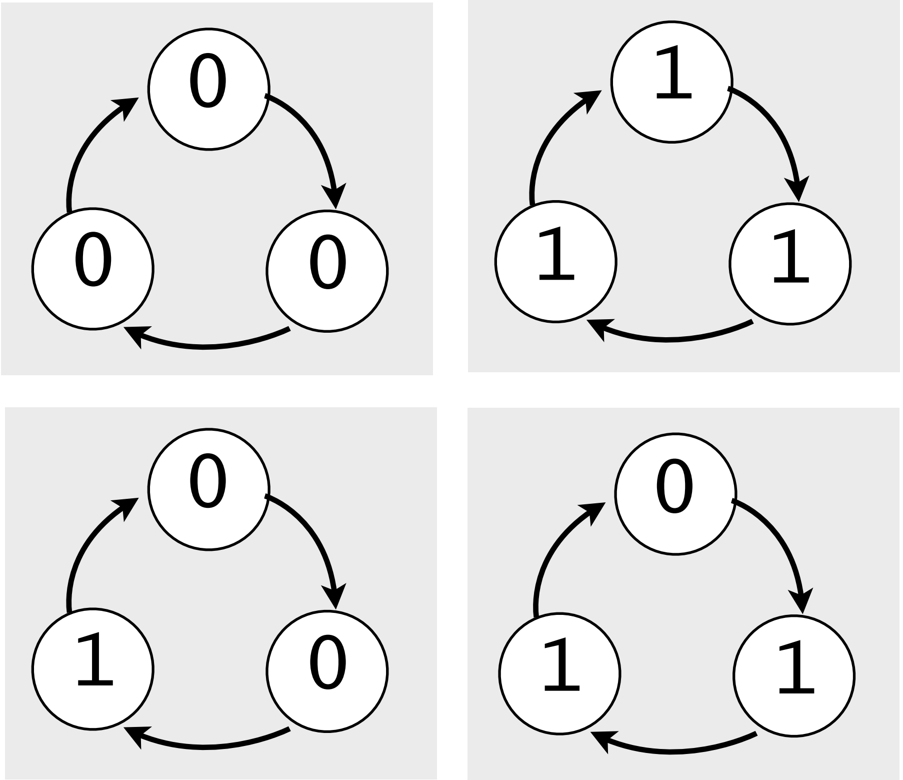
▷ **I.1.** *Necklaces.* How many different types of necklace designs can you form with n beads, each having one of two colours, ○ and ●, where it is postulated that orientation matters? Here are the possibilities for $n = 1, 2, 3$,



This is equivalent to enumerating circular arrangements of two letters and an exhaustive listing program can be based on the smallest lexicographical representation of each word, as suggested by (20), p. 26. The counting sequence starts as 2, 3, 4, 6, 8, 14, 20, 36, 60, 108, 188, 352 and constitutes *EIS* **A000031**. [An explicit formula appears later in this chapter (p. 64).] What if two necklace designs that are mirror images of one another are identified? ◁



$C_3 = 4$

[2]Throughout this book, a reference such *EIS* **Axxx** points to Sloane's *Encyclopedia of Integer Sequences* [543]. The database contains more than 100 000 entries.

I. 2.2. The admissibility theorem for ordinary generating functions. This section is a formal treatment of admissibility proofs for the constructions that we have introduced. The final implication is that any specification of a constructible class translates directly into generating function equations. The translation of the cycle construction involves the Euler totient function $\varphi(k)$ defined as the number of integers in $[1, k]$ that are relatively prime to k (Appendix A.1: *Arithmetical functions*, p. 721).

Theorem I.1 (Basic admissibility, unlabelled universe)**.** *The constructions of union, cartesian product, sequence, powerset, multiset, and cycle are all admissible. The associated operators are as follows.*

Sum: $\quad \mathcal{A} = \mathcal{B} + \mathcal{C} \quad \Longrightarrow \quad A(z) = B(z) + C(z)$

Cartesian product: $\quad \mathcal{A} = \mathcal{B} \times \mathcal{C} \quad \Longrightarrow \quad A(z) = B(z) \cdot C(z)$

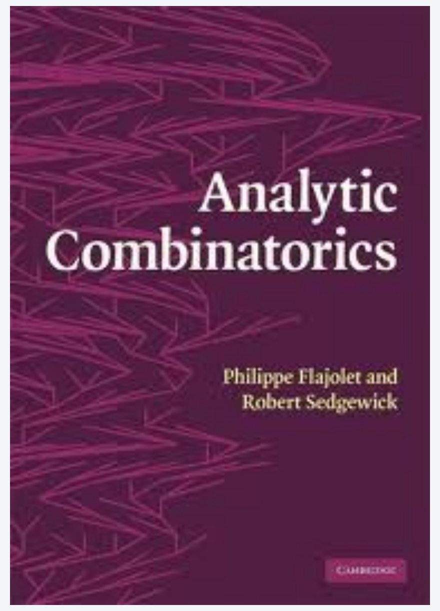
Sequence: $\quad \mathcal{A} = \mathrm{SEQ}(\mathcal{B}) \quad \Longrightarrow \quad A(z) = \dfrac{1}{1 - B(z)}$

Powerset: $\quad \mathcal{A} = \mathrm{PSET}(\mathcal{B}) \quad \Longrightarrow \quad A(z) = \begin{cases} \displaystyle\prod_{n \geq 1} (1 + z^n)^{B_n} \\ \displaystyle\exp\left(\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} B(z^k)\right) \end{cases}$

Multiset: $\quad \mathcal{A} = \mathrm{MSET}(\mathcal{B}) \quad \Longrightarrow \quad A(z) = \begin{cases} \displaystyle\prod_{n \geq 1} (1 - z^n)^{-B_n} \\ \displaystyle\exp\left(\sum_{k=1}^{\infty} \frac{1}{k} B(z^k)\right) \end{cases}$

Cycle: $\quad \mathcal{A} = \mathrm{CYC}(\mathcal{B}) \quad \Longrightarrow \quad A(z) = \displaystyle\sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \log \frac{1}{1 - B(z^k)}.$

For the sequence, powerset, multiset, and cycle translations, it is assumed that $\mathcal{B}_0 = \emptyset$.



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Circular words (necklaces). Let $\mathcal{A}$ be a binary alphabet, viewed as comprised of beads of two distinct colours. The class of *circular words* or *necklaces* (Note I.1, p. 18, and Equation (20), p. 26) is defined by a CYC composition:

$$(64) \qquad \mathcal{N} = \mathrm{CYC}(\mathcal{A}) \quad \Longrightarrow \quad N(z) = \sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \log \frac{1}{1 - 2z^k}.$$

The series starts as (*EIS* **A000031**)

$$N(z) = 2z + 3z^2 + 4z^3 + 6z^4 + 8z^5 + 14z^6 + 20z^7 + 36z^8 + 60z^9 + \cdots,$$

and the OGF can be expanded:

$$(65) \qquad N_n = \frac{1}{n} \sum_{k \mid n} \varphi(k) 2^{n/k}.$$

It turns out that $N_n = D_n + 1$ where D_n is the wheel count, p. 47. [The connection is easily explained combinatorially: start from a wheel and repaint in white all the nodes that are not on the basic circle; then fold them onto the circle.] The same argument proves that the number of necklaces over an m–ary alphabet is obtained by replacing 2 by m in (65).

k	1	2	3	4	5	6	7	8	9
$\varphi(k)$	1	1	2	2	4	2	6	4	6

$$\ln\frac{1}{1-2z} = \frac{2z}{1} + \frac{(2z)^2}{2} + \frac{(2z)^3}{3} + \frac{(2z)^4}{4} + \frac{(2z)^5}{5} + \ldots$$

$$\ln\frac{1}{1-2z^2} = \frac{2z^2}{1} + \frac{(2z^2)^2}{2} + \ldots$$

$$\ln\frac{1}{1-2z^3} = \frac{2z^3}{1} + \frac{(2z^3)^2}{2} + \ldots$$

$$\ln\frac{1}{1-2z^4} = \frac{2z^4}{1} + \ldots$$

$$\ln\frac{1}{1-2z^5} = \frac{2z^5}{1} + \ldots$$

Circular words (necklaces). Let $\mathcal{A}$ be a binary alphabet, viewed as comprised of beads of two distinct colours. The class of *circular words* or *necklaces* (Note I.1, p. 18, and Equation (20), p. 26) is defined by a CYC composition:

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AC Basics Q&A 2

Q. How many cyclic bitstrings of length 10 ?

construction

OGF $N(z) =$

result $[z^{10}]N(z) =$

$$A(z) = \sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \log \frac{1}{1 - B(z^k)}.$$

AC Basics Q&A 3

Q. How many *unimodal* permutations of length *n* ?

▷ **I.2.** *Unimodal permutations.* Such a permutation has exactly one local maximum. In other words it is of the form $\sigma_1 \cdots \sigma_n$ with $\sigma_1 < \sigma_2 < \cdots < \sigma_k = n$ and $\sigma_k = n > \sigma_{k+1} > \cdots > \sigma_n$, for some $k \geq 1$. How many such permutations are there of size n? For $n = 5$, the number is 16: the permutations are 12345, 12354, 12453, 12543, 13452, 13542, 14532 and 15432 and their reversals. [Due to Jon Perry, see *EIS* **A000079**.] ◁

definitions	*I = increasing perms*	*D = decreasing perms*	*U = unimodal perms*
constructions	$I = SET(Z)$	$D =$	$U =$
EGFs	$I(z) = e^z$	$D(z) =$	$U(z) =$
exact result	$U_n =$		

AC Basics Q&A 4

Q. Mappings with every character appearing 2 or 0 times.

constructions

$$C = Z + Z \star SET_2(C) \qquad Y = CYC(Z \star C) \qquad M = SET(Y)$$

EGF equations

$$C(z) = z + \frac{1}{2} zC(z)^2 \qquad Y(z) = \ln \frac{1}{1 - zC(z)} \qquad M(z) = \frac{1}{1 - zC(z)}$$

EGFs

$$C(z) = \frac{1 - \sqrt{1 - 2z^2}}{z}$$

$$M(z) = \frac{1}{\sqrt{1 - 2z^2}}$$

~-approximation

$$(2n)![z^{2n}]M(z) \sim (2n)! \frac{2^n}{\sqrt{\pi n}} \quad \sim 2\left(\frac{2n}{e}\right)^{2n}$$

$$[z^n]\frac{f(z)}{(1 - z/\rho)^\alpha} \sim \frac{f(\rho)}{\Gamma(\alpha)} \rho^{-n} n^{\alpha - 1}$$

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$