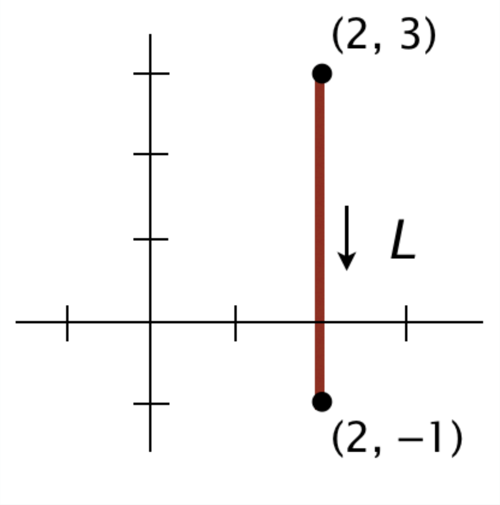
Errata group 1: Complex integration

Complex integration

Starting point:
Change variables to
convert to real
integrals.



$$\int_L z dz = \int_3^{-1} (2+iy) i dy \qquad z = x + iy \quad dz = idy$$

$$= 2i - \frac{y^2}{2}\Big|_3^{-1} = 2i + 4$$

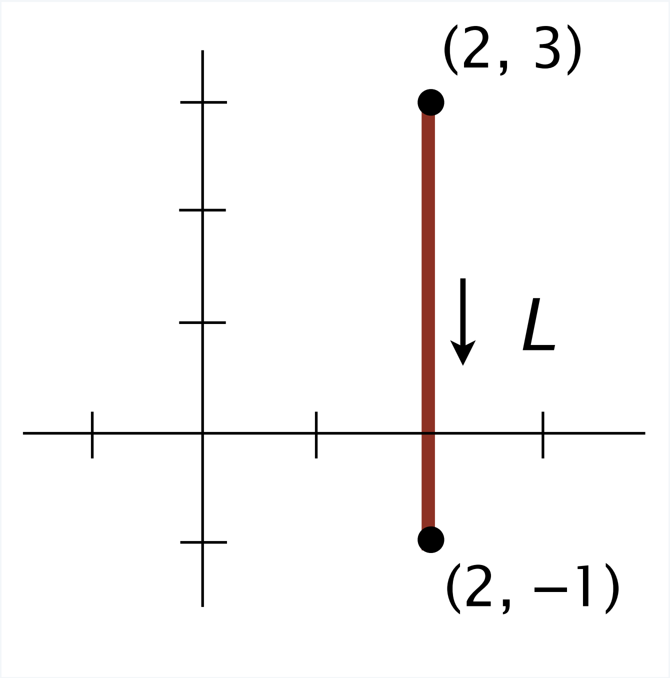
Augustin-Louis Cauchy
1789–1857

Amazing facts:

- *The integral of an analytic function around a loop is 0.*
- *The coefficients of an analytic function can be extracted via complex integration*

Analytic combinatorics context: *Immediately* gives exponential growth for meromorphic GFs

Errata group 1: Complex integration [correction]



$$z = x + iy \quad dz = idy$$

$$\int_L z\,dz = \int_3^{-1} (2 + iy)\,i\,dy \;=\; \left(2iy - \frac{y^2}{2}\right)\Big|_3^{-1} = -8i + 4$$

Much better approach. Complex integration works exactly as expected.

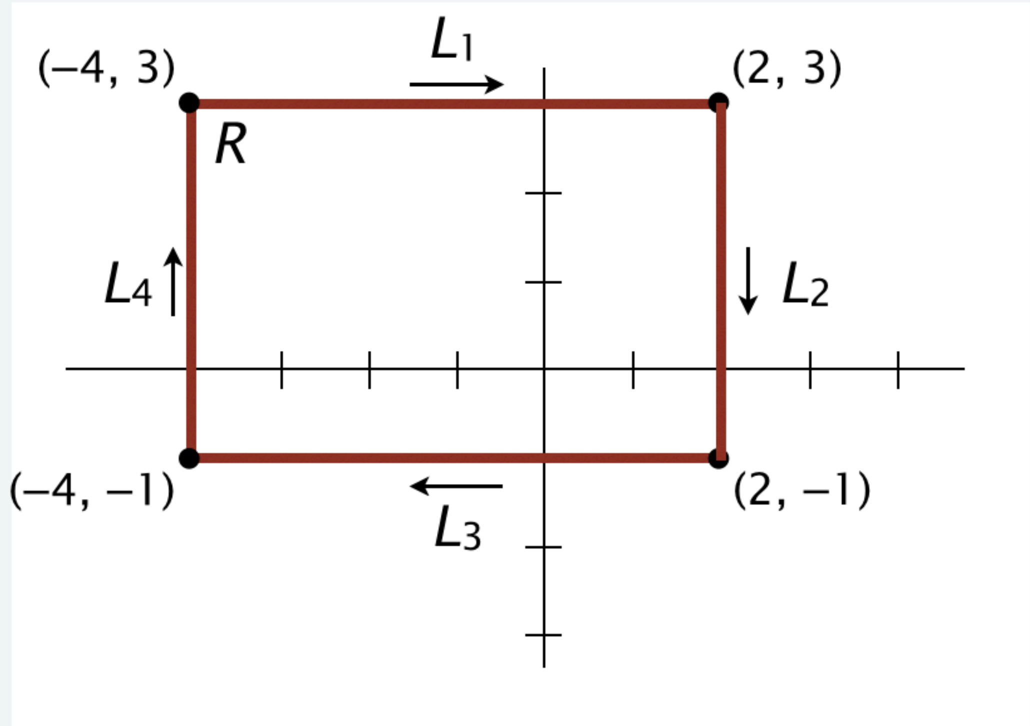
Theorem. (cf. Stein, Theorem 3.2) *If a continuous function f has an antiderivative F and* γ *is a curve from* w_1 *to* w_2 *then* $\displaystyle\int_\gamma f(z)dz = F(w_2) - F(w_1)$

$$\int_L z\,dz = \frac{z^2}{2}\Big|_{2+3i}^{2-i} \;=\; \frac{1}{2}\left((2-i)^2 - (2+3i)^2\right) \;=\; -8i + 4$$

Errata group 1: Complex integration (continued)

Integration examples

Ex 1. Integrate $f(z) = z$ on a rectangle



$$\int_{L_1} zdz = \int_{-4}^{2} xdx + 3i = \frac{x^2}{2}\Big|_{-4}^{2} + 3i = -6 + 3i$$

$$\int_{L_2} zdz = \int_{3}^{-1} (2 + iy)idy = 2i - \frac{y^2}{2}\Big|_{3}^{-1} = 2i + 4$$

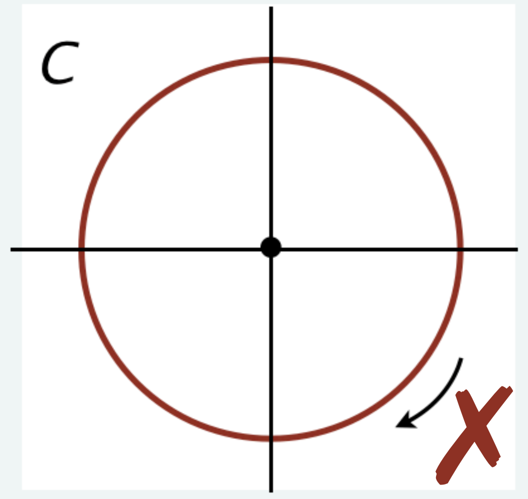
$z = x + iy \quad dz = idy$

$$\int_{L_3} zdz = \int_{2}^{-4} xdx - i = \frac{x^2}{2}\Big|_{2}^{-4} - i = 6 - i$$

$$\int_{L_4} zdz = \int_{-1}^{3} (-4 + iy)idy = -4i - \frac{y^2}{2}\Big|_{-1}^{3} = -4i - 4$$

$$\int_R zdz = \int_{L_1+L_2+L_3+L_4} zdz = -6 + 3i + 2i + 4 + 6 - i - 4i - 4 = 0 \quad (!)$$

$z = re^{i\theta} \quad dz = ire^{i\theta}d\theta$



Ex 2. Integrate $f(z) = z$ on a circle centered at 0

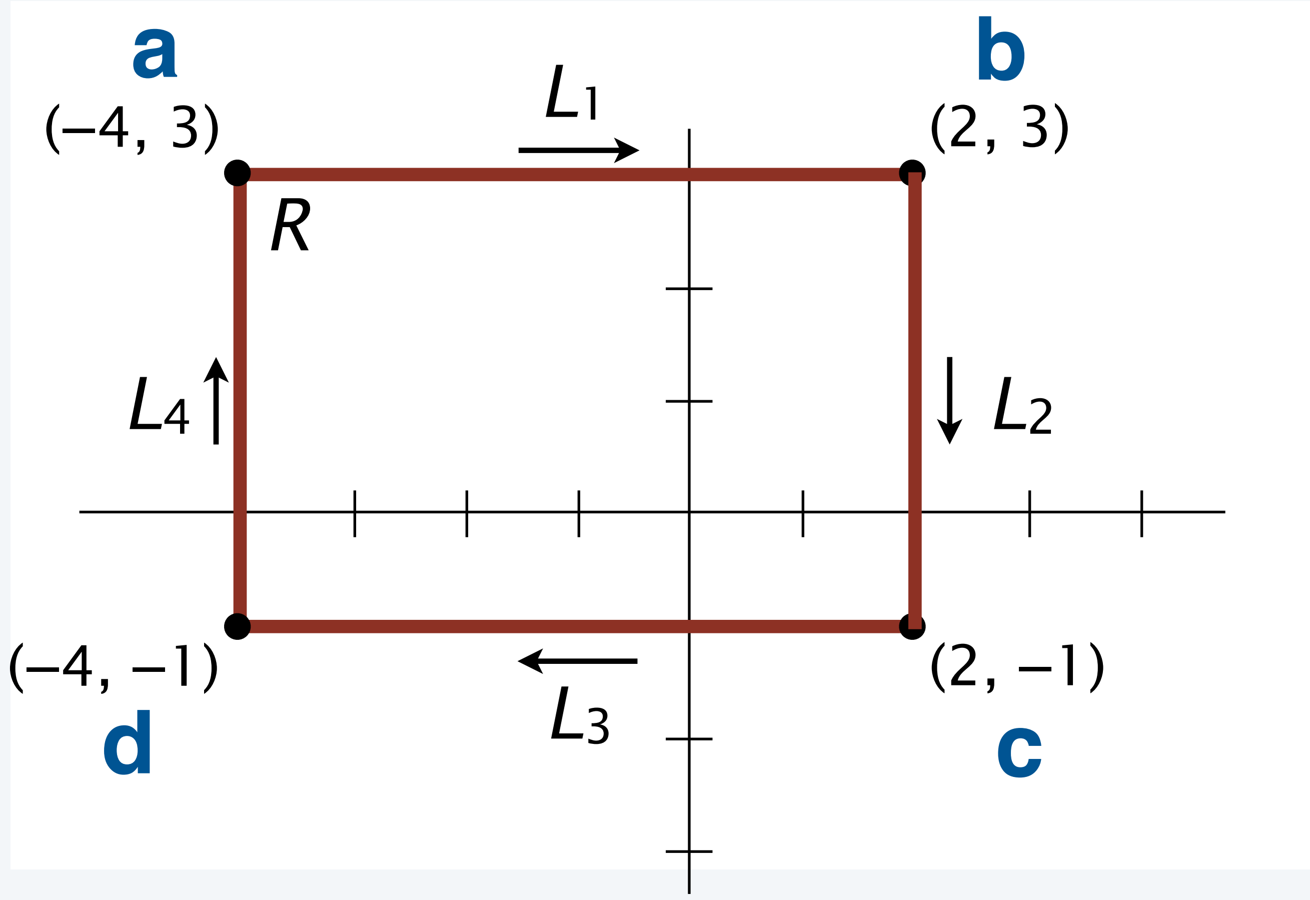
$$\int_C zdz = ir^2 \int_0^{2\pi} e^{2i\theta} d\theta = \frac{e^{2i\theta}}{2i}\Big|_0^{2\pi} = \frac{1}{2i}(1 - 1) = 0$$

Ex 3. Integrate $f(z) = 1/z$ on a circle centered at 0

$$\int_C \frac{dz}{z} = i\int_0^{2\pi} d\theta = 2\pi i$$

Errata group 1: Complex integration [improved]

Ex 1. Integrate $f(z) = z$ on a rectangle



$$\int_{L_1} z dz = \frac{z^2}{2}\bigg|_a^b = \frac{b^2}{2} - \frac{a^2}{2}$$

$$\int_{L_2} z dz = \frac{z^2}{2}\bigg|_b^c = \frac{c^2}{2} - \frac{b^2}{2}$$

$$\int_{L_3} z dz = \frac{z^2}{2}\bigg|_c^d = \frac{d^2}{2} - \frac{c^2}{2}$$

$$\int_{L_4} z dz = \frac{z^2}{2}\bigg|_d^a = \frac{a^2}{2} - \frac{d^2}{2}$$

$$\int_R z dz = \int_{L_1+L_2+L_3+L_4} z dz = 0$$

A new question was posted by Eric Neyman (April 2017).

Residues Calculated Incorrectly

I believe that every residue in the lectures is calculated incorrectly (off by a sign). The formula given for the residue is $$-\frac{f(\alpha)}{g'(\alpha)}$$, but this is incorrect. It should be $$\frac{f(\alpha)}{g'(\alpha)}$$. The error seems to be traceable back to Slide 59 of the Poles lecture slides, where instead of writing $$\frac{h_{-1}}{z - \alpha}$$, $$\frac{h_{-1}}{\alpha - z}$$ was written. But in fact the way residues are defined in Slide 51 (as in $$\frac{h_{-1}}{z - z_0}$$). I think that the reason that the asymptotics in the examples are right is because the constant $$c$$, which is claimed to be $$\frac{h_{-1}}{\alpha}$$, really should be $$(-1)^M \frac{h_{-1}}{\alpha}$$, and that these two mistakes cancel out.

Errata group 2: Residues

AC transfer theorem for meromorphic GFs (leading term)

Theorem. Suppose that $h(z)= f(z)/g(z)$ is meromorphic in $|z| \le R$ and analytic both at $z = 0$ and at all points $|z| = R$. If α is a unique closest pole to the origin of $h(z)$ in R, then α is real and $[z^N]\dfrac{f(z)}{g(z)} \sim c\beta^N N^{M-1}$ where M is the order of α, $c = (-1)^M \dfrac{Mf(\alpha)}{\alpha^M g^{(M)}(\alpha)}$ and $\beta = 1/\alpha$.

Proof sketch for $M = 1$:

- Series expansion (valid near α): $h(z) = \dfrac{h_{-1}}{\alpha - z} + h_0 + h_1(\alpha - z) + h_2(\alpha - z)^2 + \ldots$
- One way to calculate constant: $h_{-1} = \lim\limits_{z\to\alpha}(\alpha - z)h(z)$
- Approximation at α: $h(z) \sim \dfrac{h_{-1}}{\alpha - z} = \dfrac{1}{\alpha}\dfrac{h_{-1}}{1 - z/\alpha} = \dfrac{h_{-1}}{\alpha}\sum\limits_{N\ge 0}\dfrac{z^N}{\alpha^N}$

elementary from Pringsheim's and coefficient extraction theorems

✗ should be z-α everywhere

leads to numerous sign errors in later slides

similar error on p. 256 in book

See next slide for calculation of c and $M > 1$.

Notes:

- Error is *exponentially small* (and next term may involve periodicities due to complex roots).
- Result is the same as for rational functions.

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Bottom line 1

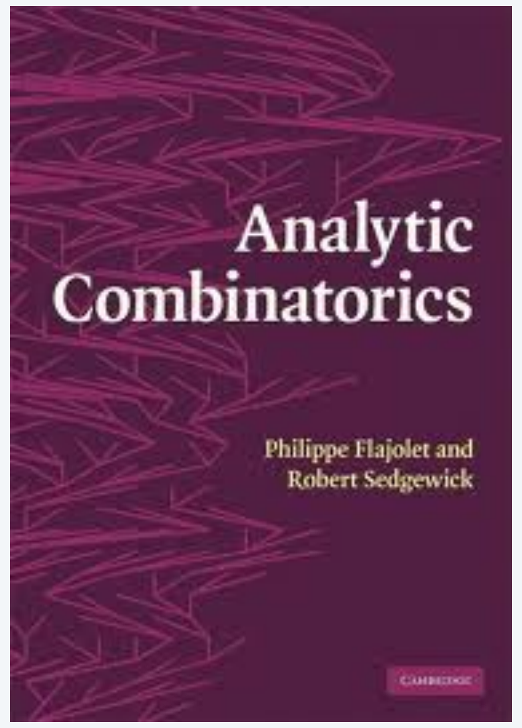
✗ should be -n-r

✗ should be α_1-z

▷ **IV.26.** *A simple exercise.* Let $f(z)$ be as in Theorem IV.9, assuming additionally a single dominant pole α_1, with multiplicity r. Then, by inspection of the proof of Theorem IV.9:

$$f_n = \frac{C}{(r-1)!}\alpha_1^{-n+r} n^{r-1}\left(1 + O\left(\frac{1}{n}\right)\right) \qquad \text{with} \quad C = \lim_{z\to\alpha_1}(z-\alpha_1)^r f(z).$$

This is certainly the most direct illustration of the Second Principle: under the assumptions, a one-term asymptotic expansion of the function at its dominant singularity suffices to determine the asymptotic form of the coefficients. ◁



$f(z)$ rational with a single dominant pole α

$$[z^N]f(z) = \frac{\beta^N N^{M-1}}{(M-1)!\alpha^M}\lim_{z\to\alpha}(\alpha - z)^M f(z)$$

where $\beta = 1/\alpha$ and M is the multiplicity of α

Bottom line 2

$h(z)$ meromorphic with a single dominant pole α

$$[z^N]h(z) = \frac{(-1)^M M f(\alpha)}{\alpha^M g^{(M)}(\alpha)} \beta^N N^{M-1}$$

where $\beta = 1/\alpha$ and M is the multiplicity of α