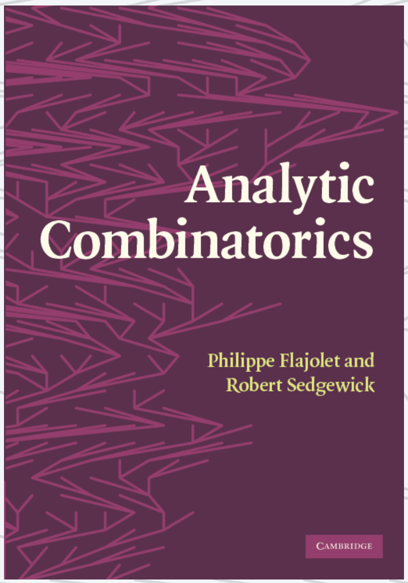

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5. Applications of Rational and Meromorphic Asymptotics

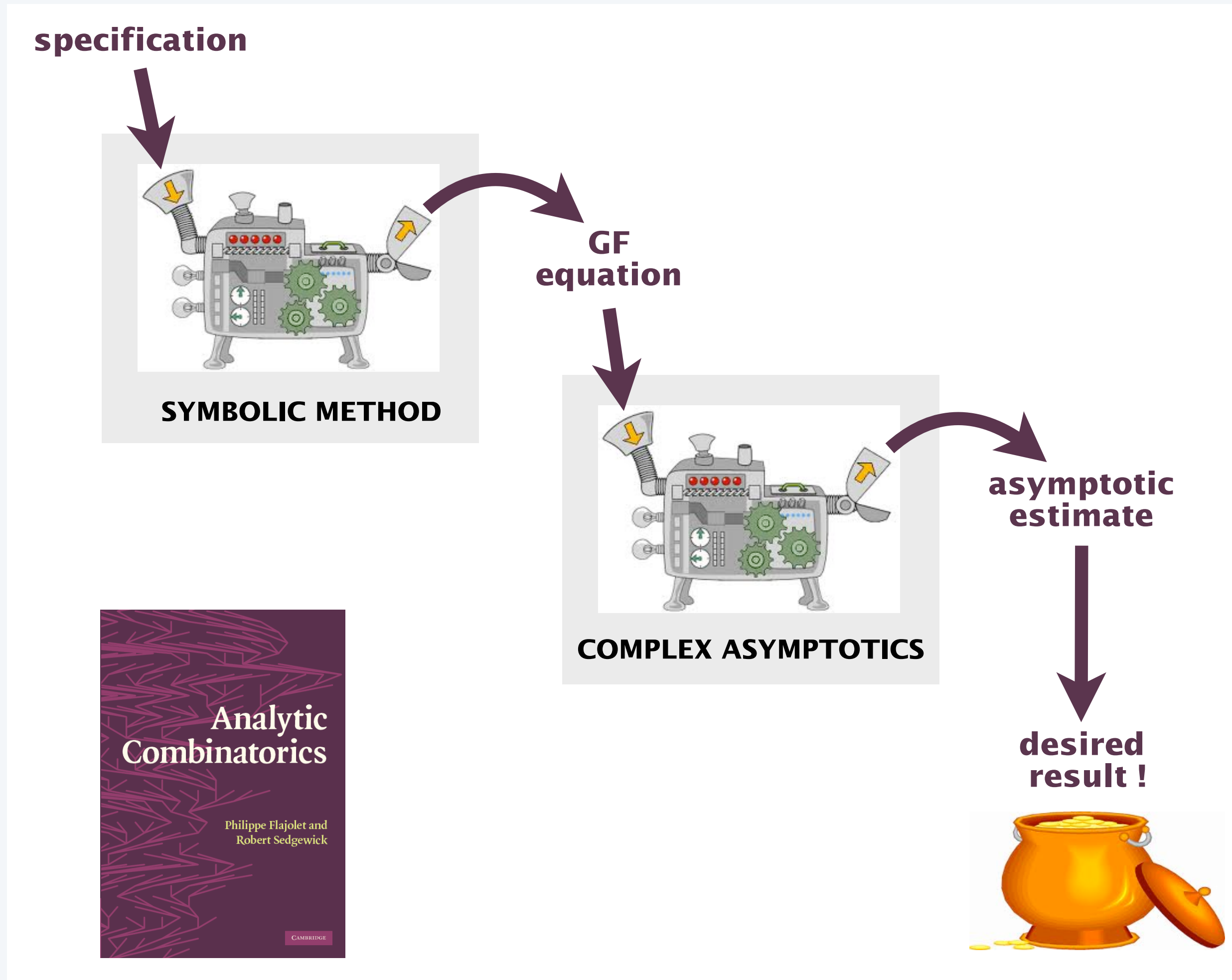
Analytic combinatorics overview

A. SYMBOLIC METHOD

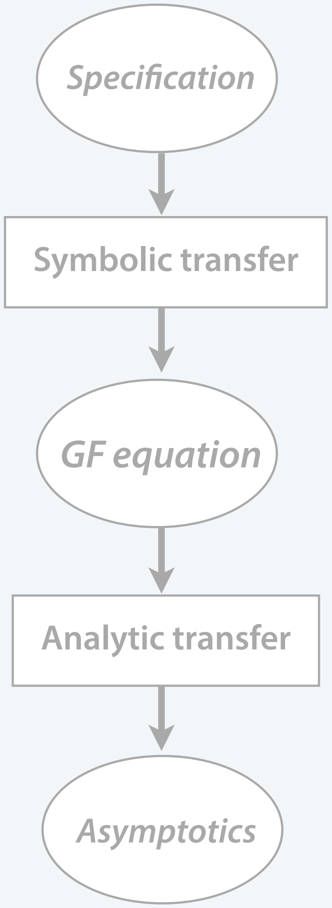
1. OGFs
2. EGFs
3. MGFs

B. COMPLEX ASYMPTOTICS

4. Rational & Meromorphic
5. Applications of R&M
6. Singularity Analysis
7. Applications of SA
8. Saddle point



Bottom line from last lecture

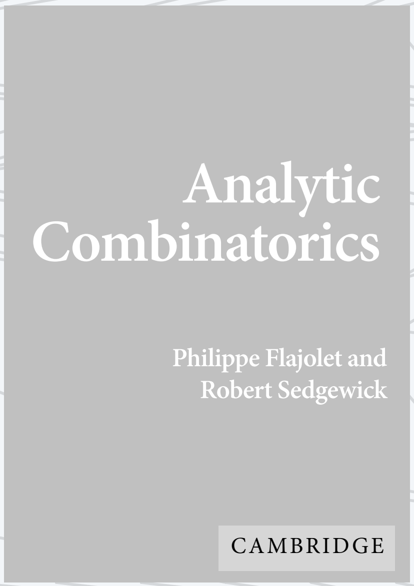


Analytic transfer for meromorphic GFs: $f(z)/g(z) \sim c\,\beta^N$

- Compute the dominant pole α (smallest real with $g(z) = 0$).
- Compute the residue $h_{-1} = -f(\alpha)/g'(\alpha)$.
- Constant c is $h_{-1}\,/\alpha$.
- Exponential growth factor β is $1/\alpha$

Not order 1 if $g'(\alpha) = 0$.
Adjust to (slightly) more complicated order M case.

This lecture: Numerous applications



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5. Applications of Rational and Meromorphic Asymptotics

- **Bitstrings**
- Other familiar examples
- Compositions
- Supercritical sequence schema

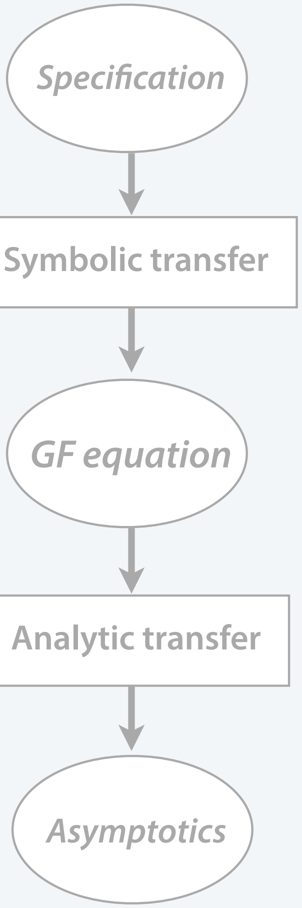
Warmup: Bitstrings

How many bitstrings of length N?

$B_0 = 1$	$B_1 = 2$	$B_2 = 4$	$B_3 = 8$	$B_4 = 16$
	0	0 0	0 0 0	0 0 0 0
	1	0 1	0 0 1	0 0 0 1
		1 0	0 1 0	0 0 1 0
		1 1	0 1 1	0 0 1 1
			1 0 0	0 1 0 0
			1 0 1	0 1 0 1
			1 1 0	0 1 1 0
			1 1 1	0 1 1 1
				1 0 0 0
				1 0 0 1
				1 0 1 0
				1 0 1 1
				1 1 0 0
				1 1 0 1
				1 1 1 0
				1 1 1 1

counting sequence	*OGF*
$B_N = 2^N$	$\dfrac{1}{1-2z}$

$$\sum_{N\geq 0} 2^N z^N = \sum_{N\geq 0} (2z)^N = \frac{1}{1-2z}$$



B, the class of all bitstrings

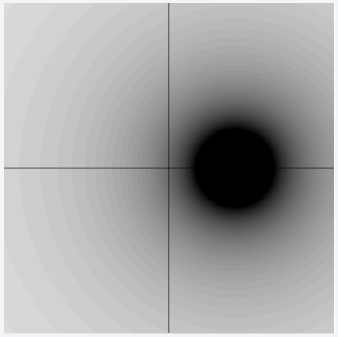
$$\mathbf{B} = \mathbf{E} + (\mathbf{Z}_0 + \mathbf{Z}_1) \times \mathbf{B}$$

$$\downarrow$$

$$B(z) = \frac{1}{1-2z}$$

$$\downarrow$$

$$[z^N]B(z) = 2^N$$



Dominant singularity: *pole* at $\alpha = 1/2$

Residue: $h_{-1} = -\dfrac{f(z)}{g'(z)} = \dfrac{1}{2}$

Coefficient of z^N: $\sim \dfrac{h_{-1}}{\alpha}\left(\dfrac{1}{\alpha}\right)^N = 2^N$

Example 1: Bitstrings with restrictions on consecutive 0s

How many bitstrings of length N have no two consecutive 0s ?

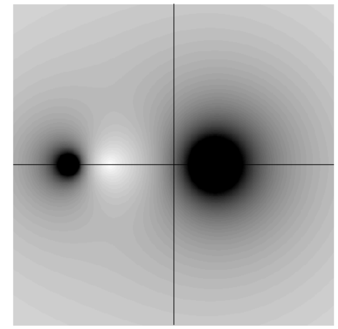
$T_0 = 1$	$T_1 = 2$	$T_2 = 3$	$T_3 = 5$	$T_4 = 8$	$T_5 = 13$
	0	0 1	0 1 1	0 1 0 1	0 1 0 1 1
	1	1 0	0 1 0	0 1 1 0	0 1 0 1 0
		1 1	1 0 1	0 1 1 1	0 1 1 0 1
			1 1 0	1 0 1 1	0 1 1 1 0
			1 1 1	1 0 1 0	0 1 1 1 1
				1 1 0 1	1 0 1 0 1
				1 1 1 0	1 0 1 1 0
				1 1 1 1	1 0 1 1 1
					1 1 0 1 1
					1 1 0 1 0
					1 1 1 0 1
					1 1 1 1 0
					1 1 1 1 1

Example 1: Bitstrings with restrictions on consecutive 0s

Specification → Symbolic transfer → GF equation → Analytic transfer → Asymptotics

$\boldsymbol{B}_{00}$, the class of all bitstrings having no 00

$$\boldsymbol{B}_{00} = \boldsymbol{E} + \boldsymbol{Z}_0 + (\boldsymbol{Z}_0 + \boldsymbol{Z}_0 \times \boldsymbol{Z}_1) \times \boldsymbol{B}_{00}$$



$$\hat{\phi} = \frac{\sqrt{5}-1}{2}$$

$$\phi = \frac{\sqrt{5}+1}{2}$$

$$B_{00}(z) = \frac{1+z}{1-z-z^2}$$

Dominant singularity: *pole* at $\hat{\phi}$

Residue: $h_{-1} = -\frac{f(\hat{\phi})}{g'(\hat{\phi})} = \frac{1+\hat{\phi}}{1+2\hat{\phi}}$

Coefficient of z^N: $\sim \frac{h_{-1}}{\hat{\phi}}\left(\frac{1}{\hat{\phi}}\right)^N = \frac{1+\hat{\phi}}{\hat{\phi}+2\hat{\phi}^2}\phi^N$

$$[z^N]B_{00}(z) = \frac{\phi^2}{\sqrt{5}}\phi^N$$

$$\sim c_2\beta_2^N \quad \text{with} \begin{cases} \beta_2 \doteq 1.61803 \\ c_2 \doteq 1.17082 \end{cases}$$

$$\phi\hat{\phi} = 1$$

$$\phi^2 = \phi + 1$$

Example 1: Bitstrings with restrictions on consecutive 0s

Specification

Symbolic transfer

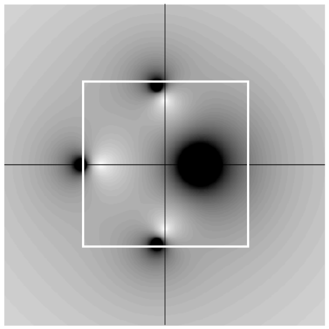
GF equation

Analytic transfer

Asymptotics

B_4, the class of all bitstrings having no 0^4

$$\mathbf{B}_4 = \mathbf{Z}_{<4} (\mathbf{E} + \mathbf{Z}_1 \mathbf{B}_4)$$



$$B_4(z) = (1 + z + z^2 + z^3)(1 + zB_4(z))$$

$$= \frac{1 + z + z^2 + z^3}{1 - z - z^2 - z^3 - z^4}$$

Dominant singularity: *pole* at α

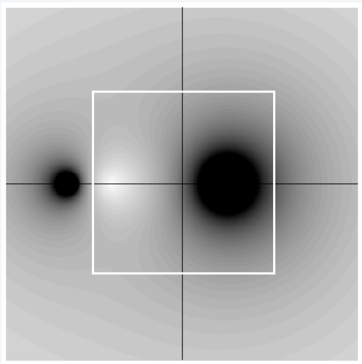
Residue: $h_{-1} = -\frac{f(z)}{g'(z)} = \frac{1 + \alpha + \alpha^2 + \alpha^3}{\alpha + 2\alpha + 3\alpha^2 + 4\alpha^3}$

Coefficient of z^N: $[z^N]B_4(z) \sim \frac{h_{-1}}{\alpha}\left(\frac{1}{\alpha}\right)^N$

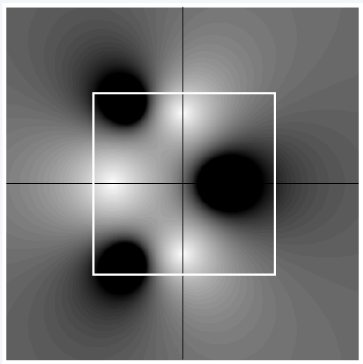
$$[z^N]B_4(z) \sim c_4 \beta_4^N \quad \text{with} \begin{cases} \beta_4 \doteq 1.9276 \\ c_4 \doteq 1.0917 \end{cases}$$

Example 1: Bitstrings with restrictions on consecutive 0s

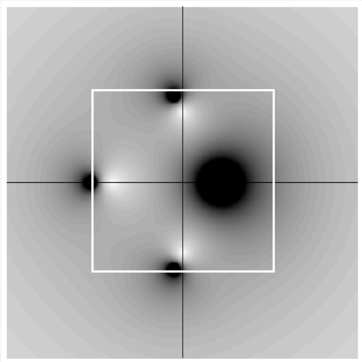
$$\frac{1+z}{1-z-z^2}$$



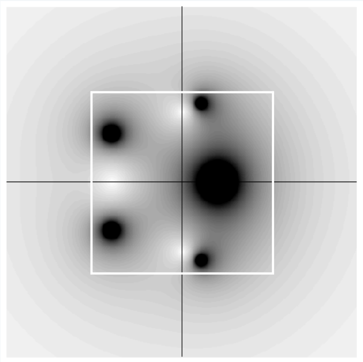
$$\frac{1+z+z^2}{1-z-z^2-z^3}$$



$$\frac{1+z+z^2+z^3}{1-z-z^2-z^3-z^4}$$

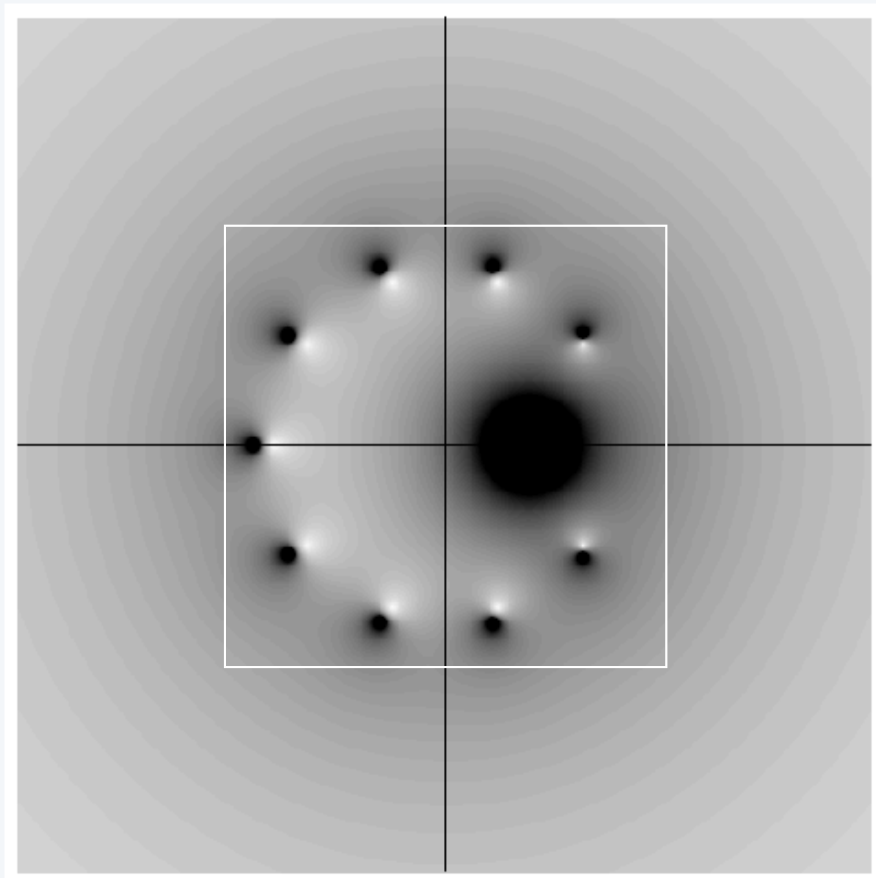


$$\frac{1+z+z^2+z^3+z^4}{1-z-z^2-z^3-z^4-z^5}$$



Example 1: Bitstrings with restrictions on consecutive 0s

$$\frac{1+z+z^2+z^3+z^4+z^5+z^6+z^7+z^8+z^9}{1-z-z^2-z^3-z^4-z^5-z^6-z^7-z^8-z^9-z^{10}}$$



Information on consecutive 0s in GFs for strings

[from AC Part I Lecture 5]

$$B_M(z) = \sum_{b \in B_M} z^{|b|} = \sum_{N \geq 0} \{\text{\# of bitstrings of length } N \text{ with no } 0^M\} z^N$$

$$= \frac{1 + z + z^2 + \ldots + z^{M-1}}{1 - z - z^2 - \ldots z^M} = \frac{1 - z^M}{1 - 2z + z^{M+1}}$$

$$B_M(z/2) = \sum_{N \geq 0} (\{\text{\# of bitstrings of length } N \text{ with no runs of } M \text{ 0s}\}/2^N) z^N$$

$$B_M(1/2) = \sum_{N \geq 0} \{\text{\# of bitstrings of length } N \text{ with no runs of } M \text{ 0s}\}/2^N$$

$$= \sum_{N \geq 0} \Pr\{\text{1st } N \text{ bits of a random bitstring have no runs of } M \text{ 0s}\}$$

$$= \sum_{N \geq 0} \Pr\{\text{position of end of first } 0^M \text{ is} > N\} = \text{Expected position of end of first } 0^M$$

Theorem. Probability that an N-bit random bitstring has no 0^M: $[z^N]B_M(z/2) \sim c_M(\beta_M/2)^N$

Theorem. Expected wait time for the first 0^M in a random bitstring: $B_M(1/2) = 2^{M+1} - 2$

Autocorrelation

[from AC Part I Lecture 5]

The probability that an N-bit random bitstring does not contain 0000 is $\sim 1.0917 \times .96328^N$

The expected wait time for the first occurrence of 0000 in a random bitstring is 30.

Q. Do the same results hold for 0001?

A. NO!

0001 occurs much earlier than 0000

10111110100101001100111**0001**0011111011011010**0000**0111100001

Observation. Consider first occurrence of 000.

- 0000 and 0001 equally likely, BUT
- mismatch for 0000 means 0001, so need to wait four more bits
- mismatch for 0001 means 0000, so *next* bit could give a match.

Q. What is the probability that an N-bit random bitstring does not contain 0001?

Q. What is the expected wait time for the first occurrence of 0001 in a random bitstring?

Constructions for strings without specified patterns

[from AC Part I Lecture 5]

Cast of characters:

p — a pattern

S_p — binary strings that do not contain p

T_p — binary strings that *end in p* *and have no other occurrence of p*

p 101001010

S_p 10111110101101001100110000011111

T_p 10111110101101001100110101001010

First construction

- S_p and T_p are disjoint
- the empty string is in S_p
- adding a bit to a string in S_p gives a string in S_p or T_p

$$S_p + T_p = E + S_p \times \{Z_0 + Z_1\}$$

Constructions for bitstrings without specified patterns

[from AC Part I Lecture 5]

Every pattern has an autocorrelation polynomial

- slide the pattern to the left over itself.
- for each match of i trailing bits with the leading bits include a term $z^{|p|-i}$

101001010	
101001010	z^0
101001010	
101001010	
101001010	
101001010	
101001010	z^5
101001010	
101001010	z^7
101001010	

autocorrelation polynomial → $c_{101001010}(z) = 1 + z^5 + z^7$

Constructions for bitstrings without specified patterns

[from AC Part I Lecture 5]

Second construction

- for each 1 bit in the autocorrelation of any string in T_p add a "tail"
- result is a string in S_p followed by the pattern

p 101001010

a string in T_p 10111110101101001100110101001010

strings in S_p

10111110101101001100110101001010

101111101011010011001101010010101001010

1011111010110100110011010100101011001010

first tail is null

101001010
101001010
101001010
101001010
101001010
101001010
1010010101001010
101001010
1010010101001010
101001010

$$S_p \times \{p\} = T_p \times \sum_{c_i \neq 0} \{t_i\}$$

Bitstrings without specified patterns

[from AC Part I Lecture 5]

How many N-bit strings do not contain a specified pattern p ?

Classes	S_p — the class of binary strings with no p
	T_p — the class of binary strings that end in p *and have no other occurence*

OGFs	$S_p(z) = \sum_{s \in S_p} z^{\lvert s \rvert}$
	$T_p(z) = \sum_{s \in T_p} z^{\lvert s \rvert}$

Constructions $\quad S_p + T_p = E + S_p \times \{Z_0 + Z_1\} \qquad S_p \times \{p\} = T_p \times \sum_{c_i \neq 0} \{t_i\}$

OGF equations $\quad S_p(z) + T_p(z) = 1 + 2zS_p(z) \qquad S_p(z)z^P = T_p(z)c_p(z)$

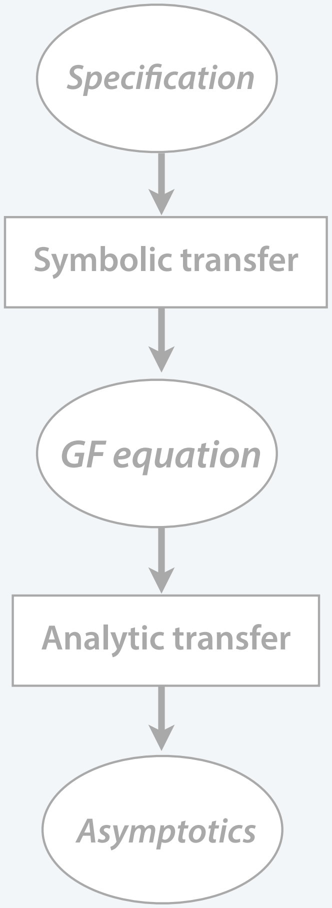
Solution

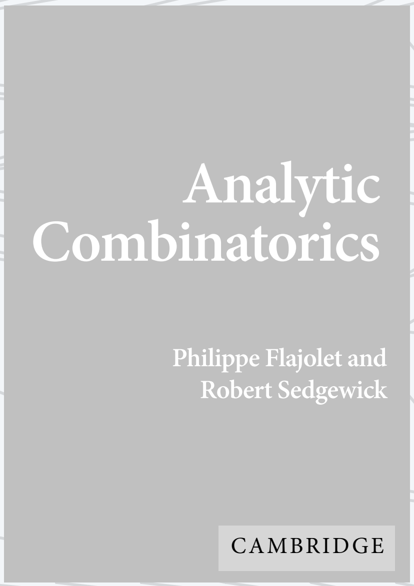
$$S_p(z) = \frac{c_p(z)}{z^P + (1 - 2z)c_p(z)}$$

Extract cofficients

$$[z^N]S_p(z) \sim c_p \beta_p^N \quad \text{where} \begin{cases} \beta_p \text{ is the dominant root of } z^P + (1-2z)c_p(z) \\ c_p = \text{ [explicit formula available]} \end{cases}$$

Bitstrings without specified patterns





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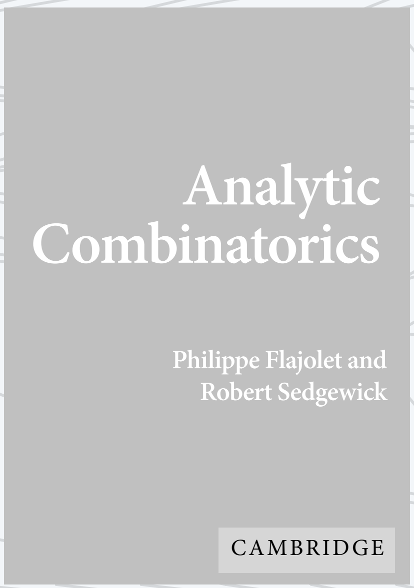
5. Applications of Rational and Meromorphic Asymptotics

- **Bitstrings**
- Other familiar examples
- Compositions
- Supercritical sequence schema



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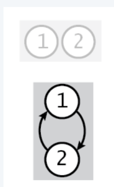
5. Applications of Rational and Meromorphic Asymptotics

- Bitstrings
- **Other familiar examples**
- Compositions
- Supercritical sequence schema

Example 2: Derangements

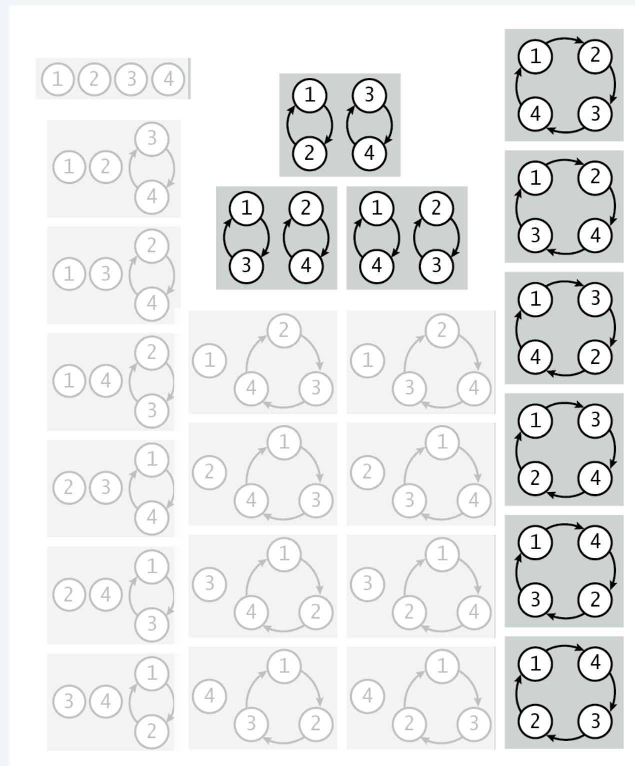
How many permutations of size N have no singleton cycles ?

$D_1 = 0$



$D_2 = 1$

$D_3 = 2$



$D_4 = 9$

Example 2: Derangements

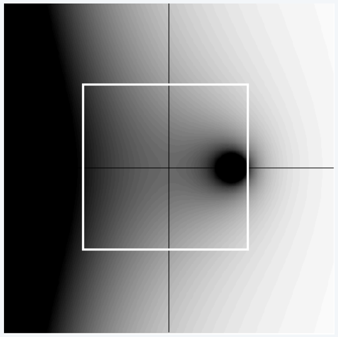
Specification → Symbolic transfer → GF equation → Analytic transfer → Asymptotics

D, the class of all permutations with no singleton cycles

D = SET(CYC$_{>1}$(***Z***))

$$D(z) = \frac{e^{-z}}{1-z}$$

$$N![z^N]D(z) \sim \frac{N!}{e}$$



Dominant singularity: *pole* at 1

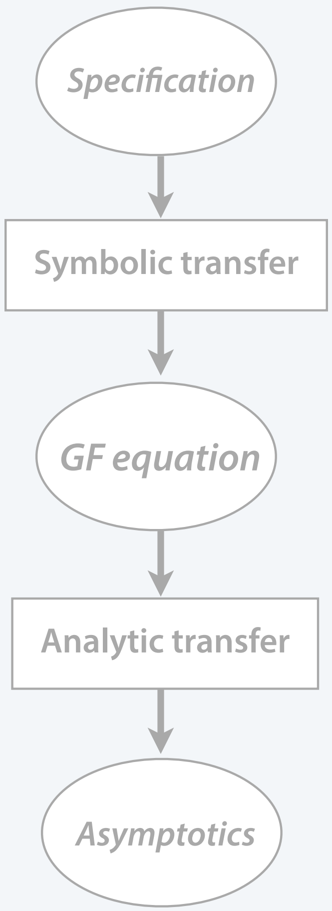
Residue: $h_{-1} = -\frac{f(1)}{g'(1)} = e^{-1}$

$$[z^N]D(z) = \frac{h_{-1}}{1}1^N = \frac{1}{e}$$

N	$N!/e$	D_N
2	.7357...	1
3	2.2072...	2
4	8.8291...	9
5	44.1455...	44

estimates are extremely accurate even for small N

Example 2: Derangements



D_M, the class of all permutations with no cycles of length $\leq M$

$$D_M = \text{SET}(\text{CYC}_{>M}(Z))$$

$$D_M(z) = \frac{e^{-z-\frac{z^2}{2}-\frac{z^3}{3}-\ldots\frac{z^M}{M}}}{1-z}$$

Dominant singularity: *pole* at 1

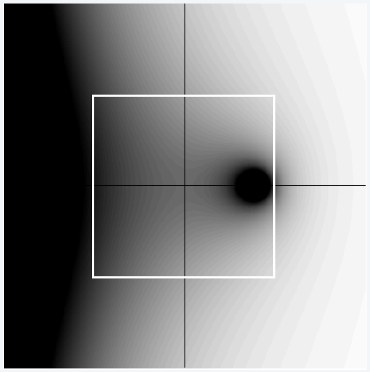
Residue: $h_{-1} = -\frac{f(1)}{g'(1)} = e^{-H_M}$

$$[z^N]D(z) = \frac{h_{-1}}{1}1^N = \frac{1}{e^{H_M}}$$

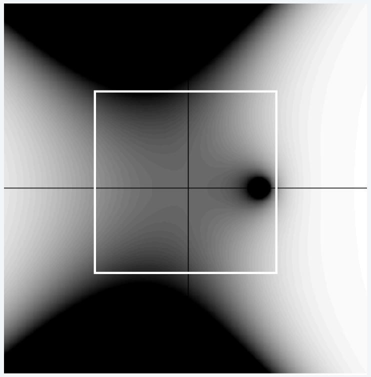
$$N![z^N]D(z) \sim \frac{N!}{e^{H_M}}$$

Example 2: Derangements

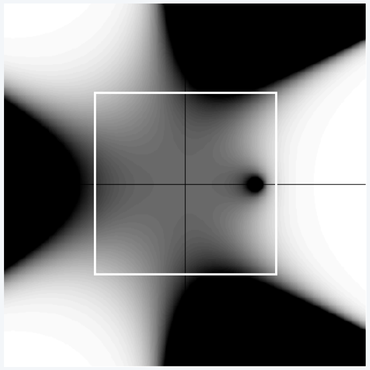
$$\frac{e^{-z}}{1-z}$$



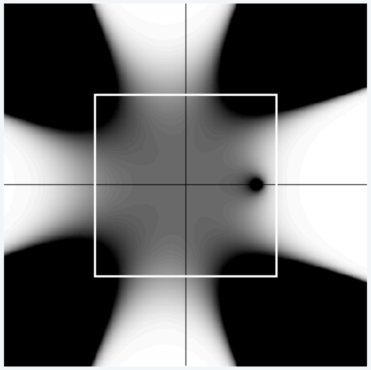
$$\frac{e^{-z-z^2/2}}{1-z}$$



$$\frac{e^{-z-z^2/2-z^3/3}}{1-z}$$

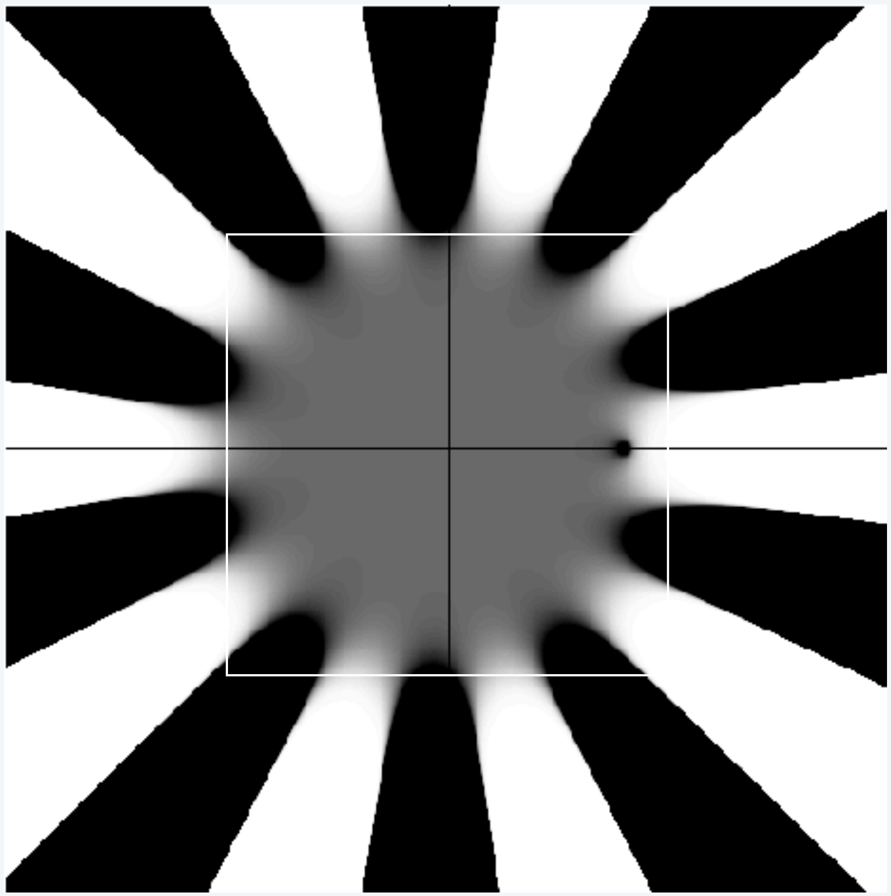


$$\frac{e^{-z-z^2/2-z^3/3-z^4/4}}{1-z}$$



Example 2: Derangements

$$\frac{e^{-z-z^2/2-z^3/3-z^4/4-z^5/5-z^6/6-z^7/7-z^8/8-z^9/9-z^{10}/10}}{1-z}$$



Example 3: Surjections

How many words of length N are M-surjections for some M ?

```
1
```

$R_1 = 1$

```
1 1
1 2
2 1
```

$R_2 = 3$

```
1 1 1
```

```
1 1 2
1 2 1
1 2 2
2 1 1
2 1 2
2 2 1
```

```
1 2 3
1 3 2
2 1 3
2 3 1
3 1 2
3 2 1
```

$R_3 = 13$

```
1 1 1 1
```

```
1 1 1 2
1 1 2 1
1 2 1 1
2 1 1 1
1 1 2 2
1 2 1 2
2 1 1 2
2 1 2 1
2 2 1 1
1 2 2 1
1 2 2 2
2 1 2 2
2 2 1 2
2 2 2 1
```

```
1 2 3 1
1 3 2 1
2 1 3 1
2 3 1 1
3 1 2 1
3 2 1 1
1 2 1 3
1 3 1 2
2 1 1 3
3 1 1 2
1 1 2 3
1 1 3 2
```

```
1 2 3 3
1 3 2 3
2 1 3 3
2 3 1 3
3 1 2 3
3 2 1 3
1 3 3 2
2 3 3 1
3 1 3 2
3 2 3 1
3 3 1 2
3 3 2 1
1 2 3 2
1 3 2 2
2 1 3 2
2 3 1 2
3 1 2 2
3 2 1 2
1 2 2 3
2 1 2 3
2 3 2 1
3 2 2 1
2 2 1 3
2 2 3 1
```

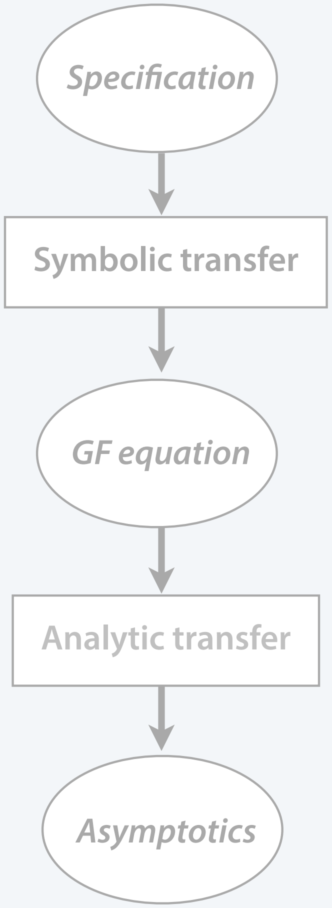
```
1 2 3 4
1 3 2 4
2 1 3 4
2 3 1 4
3 1 2 4
3 2 1 4
1 2 4 3
1 3 4 2
2 1 4 3
2 3 4 1
3 1 4 2
3 2 4 1
1 4 2 3
1 4 3 2
2 4 1 3
2 4 3 1
3 4 1 2
3 4 2 1
4 1 2 3
4 1 3 2
4 2 1 3
4 2 3 1
4 3 1 2
4 3 2 1
```

$R_4 = 75$

"coupon collector sequences"

For some M, each of the first M letters appears at least once.

Example 3: Surjections



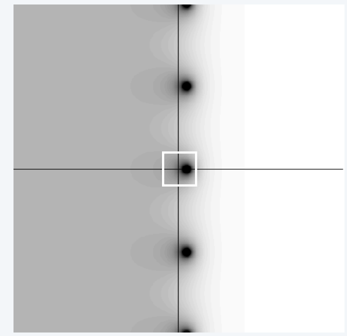
R, the class of all surjections

$$\boldsymbol{R} = \mathrm{SEQ}(\mathrm{SET}_{>0}(\boldsymbol{Z}))$$

$$R(z) = \frac{1}{1-(e^z-1)}$$

$$= \frac{1}{2-e^z}$$

$$[z^N]R(z) = \frac{1}{2(\ln 2)^{N+1}}$$



Dominant singularity: *pole* at $z = \ln 2$

Residue: $h_{-1} = -\frac{1}{g'(\ln 2)} = \frac{1}{2}$

N	$N!/2(\ln 2)^{N+1}$	R_N
2	3.0027...	3
3	12.9962...	13
4	74.9987...	75

estimates are extremely accurate
even for small N

Example 3: Surjections

How many words of length N are *double surjections* for some M ?

1 1

$R_2 = 1$

1 1 1

$R_3 = 1$

1 1 1 1	1 1 2 2
	1 2 1 2
	2 1 1 2
	2 1 2 1
	2 2 1 1
	1 2 2 1

$R_4 = 7$

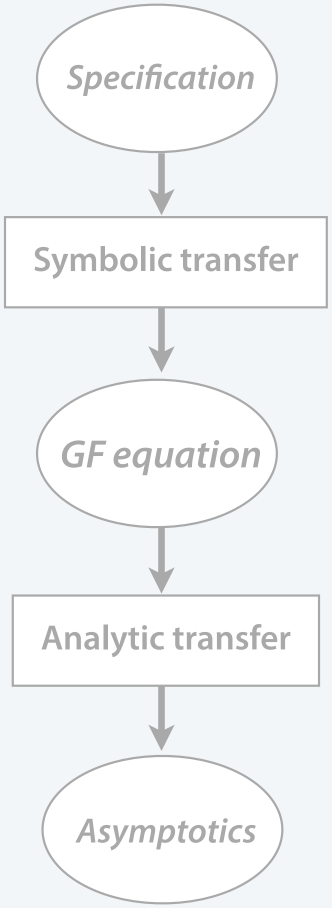
1 1 1 1 1	1 1 1 2 2	1 1 2 2 2
	1 1 2 1 2	1 2 2 1 2
	1 1 2 2 1	2 1 2 2 1
	1 2 1 1 2	2 2 1 1 2
	1 2 1 2 1	1 2 2 2 1
	1 2 2 1 1	2 2 2 1 1
	2 1 1 1 2	2 1 2 1 2
	2 1 1 2 1	1 2 1 2 2
	2 1 2 1 1	2 1 1 2 2
	2 2 1 1 1	2 2 1 2 1

$R_5 = 21$

"*double* coupon collector sequences"

For some M, each of the first M letters appears at least *twice*.

Example 3: Surjections

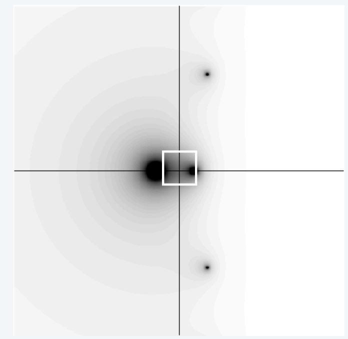


R, the class of all double surjections

$$\boldsymbol{R} = \mathrm{SEQ}(\mathrm{SET}_{>1}(\boldsymbol{Z}))$$

$$R(z) = \frac{1}{1-(e^z - z - 1)} = \frac{1}{2 + z - e^z}$$

$$R_N \sim \frac{1}{\rho + 1}\frac{N!}{\rho^{N+1}}$$



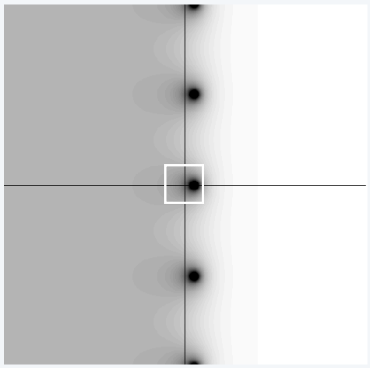
Singularities where $e^z = z + 2$

Dominant singularity: *pole* at $\rho \doteq 1.14619$

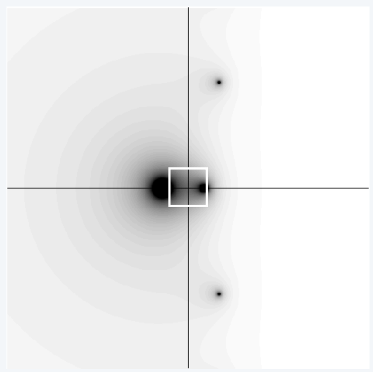
Residue: $h_{-1} = -\frac{1}{g'(\rho)} = \frac{1}{e^\rho - 1} = \frac{1}{\rho + 1}$

Example 3: Surjections

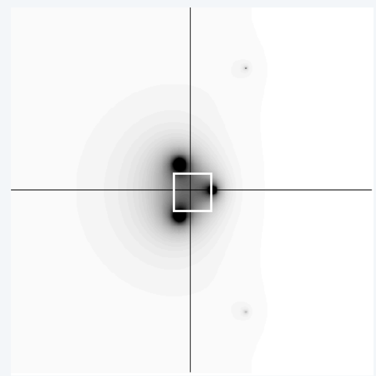
$$\frac{1}{2 - e^z}$$



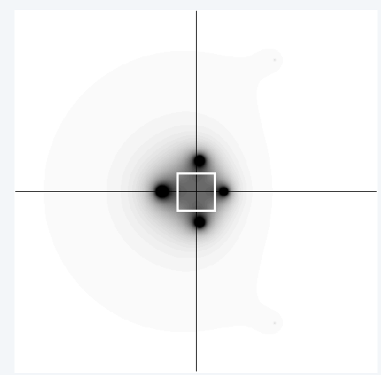
$$\frac{1}{2 + z - e^z}$$



$$\frac{1}{2 + z + z^2/2 - e^z}$$

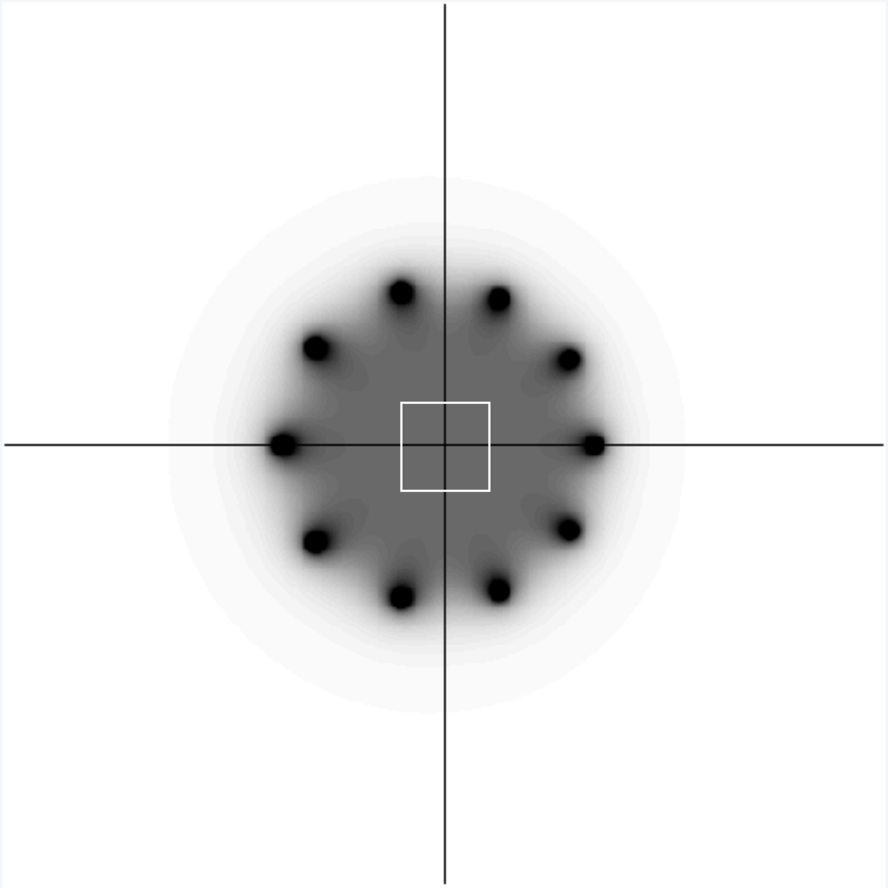


$$\frac{1}{2 + z + z^2/2 + z^3/6 - e^z}$$



Example 3: Surjections

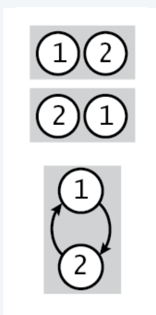
$$\frac{1}{2 + z + z^2/2 + z^3/3! + z^4/4! + z^5/5! + z^6/6! + z^7/7! + z^8/8! + z^9/9! - e^z}$$



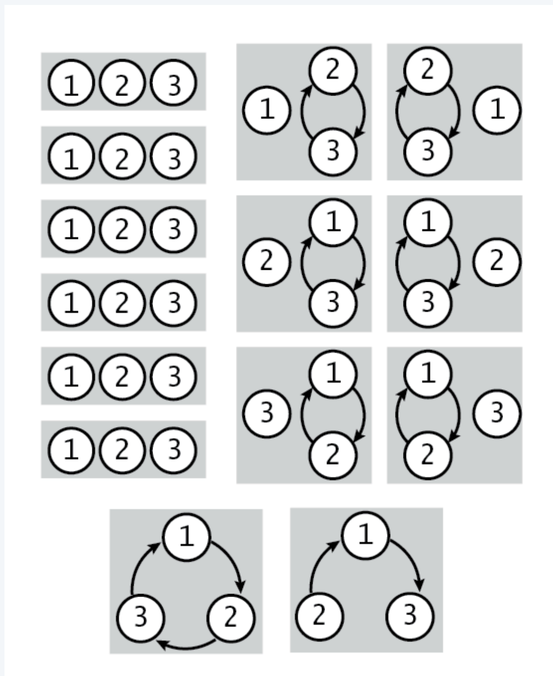
Example 4: Alignments

How many *sequences of labelled cycles* of size N?

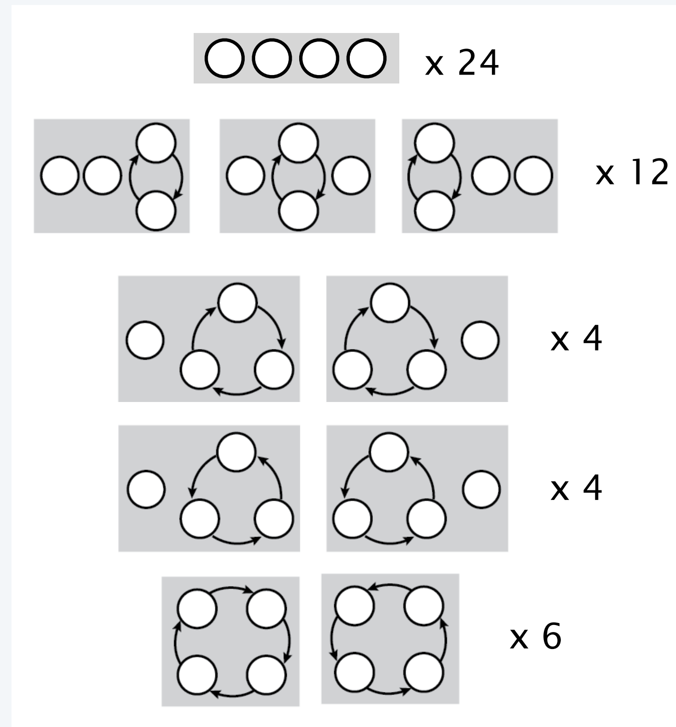
$O_1 = 1$



$O_2 = 3$

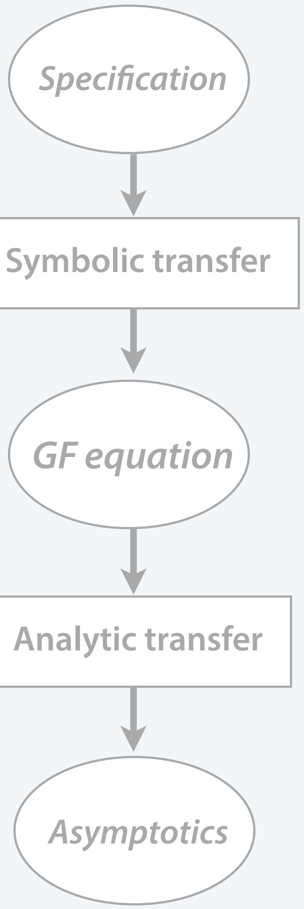


$O_3 = 14$



$O_4 = 88$

Example 3: Alignments

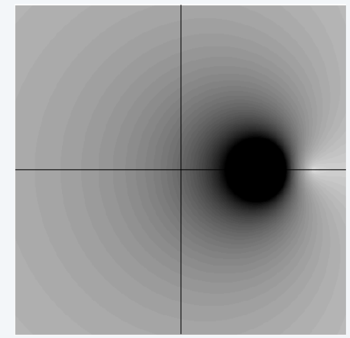


O, the class of all alignments

O = SEQ(CYC(***Z***))

$$O(z) = \frac{1}{1 - \ln \frac{1}{1-z}}$$

$$O_N \sim \frac{N!}{e(1-1/e)^{N+1}}$$



Singularities where $\ln \frac{1}{1-z} = 1$

Dominant singularity: *pole* at $z = 1 - \frac{1}{e}$

Residue: $h_{-1} = -\frac{1}{g'(1-1/e)} = \frac{1}{e}$

N	$N!/e(1-1/e)^{N+1}$	O_N
2	2.9129...	3
3	13.8247...	14
4	87.4816...	88

estimates are extremely accurate even for small N

Example 4: Set partitions

Q. How many ways to *partition an N-element set into r subsets ?* ← see Lecture 3

$S_{N2} = 2^N - 1$

only B B B... B *disallowed*

A	B	C

$S_{33} = 1$

A	B	C	C
A	B	C	B
A	B	B	C
A	B	C	A
A	A	B	C
A	B	A	C

$S_{43} = 6$

A	B	C	A	A
A	B	C	A	B
A	B	C	A	C
A	B	C	B	A
A	B	C	B	B
A	B	C	B	C
A	B	C	C	A
A	B	C	C	B
A	B	C	C	C

A	B	A	C	A
A	B	A	C	B
A	B	A	C	C
A	B	B	C	A
A	B	B	C	B
A	B	B	C	C

A	B	B	B	C
A	B	A	B	C
A	A	B	C	C
A	A	B	C	B
A	A	B	B	C
A	A	B	C	A
A	A	A	B	C
A	A	B	A	C

$S_{53} = 25$

Application: rhyming schemes

There was a small boy of Quebec **A**
Who was buried in snow to his neck **A**
When they said, "Are you friz?" **B**
He replied, " Yes, I is — **B**
But we don't call this cold in Quebec! **A**

TWO roads diverged in a yellow wood, **A**
And sorry I could not travel both **B**
And be one traveler, long I stood **A**
And looked down one as far as I could **A**
To where it bent in the undergrowth; **B**

Example 4: Set partitions

Specification

Symbolic transfer

GF equation

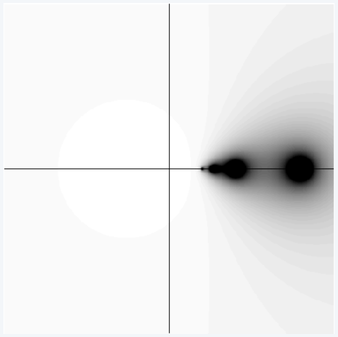
Analytic transfer

Asymptotics

S_r, the class of all poems with r rhymes

$$\mathbf{S_r} = Z_A \times SEQ(Z_A) \times Z_B \times SEQ(Z_A + Z_B) \times Z_C \times SEQ(Z_A + Z_B + Z_C) \times \ldots$$

$$S_r(z) = \frac{z^r}{(1-z)(1-2z)\ldots(1-rz)}$$



Singularities at 1, 1/2, 1/3, ... $1/r$

Dominant singularity: *pole* at $1/r$

Residue: $h_{-1} = -\frac{f(1/r)}{g'(1/r)} = \frac{1}{r \cdot r!}$

$$[z^N]S^r(z) \sim \frac{r^N}{r!}$$

Analytic Combinatorics

Philippe Flajolet and Robert Sedgewick

CAMBRIDGE

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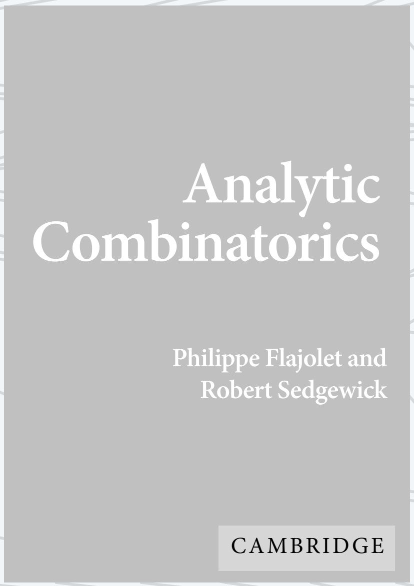
5. Applications of Rational and Meromorphic Asymptotics

- Bitstrings
- **Other familiar examples**
- Compositions
- Supercritical sequence schema



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5. Applications of Rational and Meromorphic Asymptotics

- Bitstrings
- Other familiar examples
- **Compositions**
- Supercritical sequence schema

Example 5: Compositions

Q. How many ways to express N as a sum of positive integers?

1

$I_1 = 1$

1 + 1
2

$I_2 = 2$

1 + 1 + 1
1 + 2
2 + 1
3

$I_3 = 4$

1 + 1 + 1 + 1
1 + 1 + 2
1 + 2 + 1
1 + 3
2 + 1 + 1
2 + 2
3 + 1
4

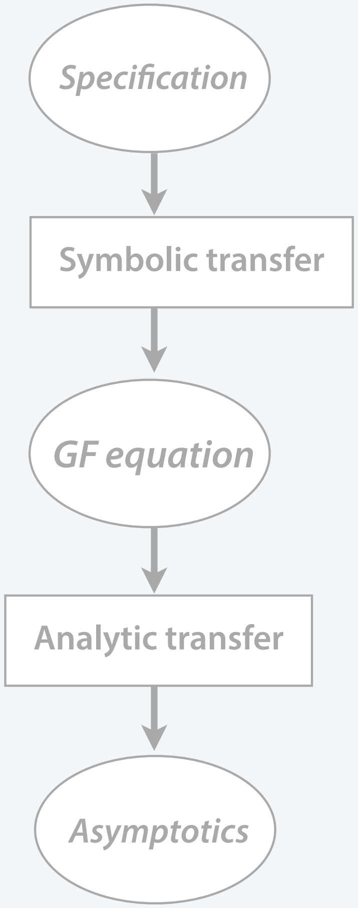
$I_4 = 8$

1 + 1 + 1 + 1 + 1
1 + 1 + 1 + 2
1 + 1 + 2 + 1
1 + 1 + 3
1 + 2 + 1 + 1
1 + 2 + 2
1 + 3 + 1
1 + 4
2 + 1 + 1 + 1
2 + 1 + 2
2 + 2 + 1
2 + 3
3 + 1 + 1
3 + 2
4 + 1
5

$I_5 = 16$

A. $I_N = 2^{N-1}$

Example 5: Compositions

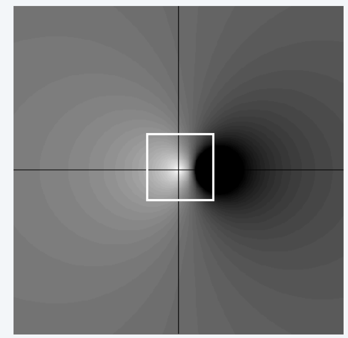


I, the class of all positive integers

$$\boldsymbol{I} = \text{SEQ}_{>0}(\boldsymbol{Z})$$

$$I(z) = \frac{z}{1-z}$$

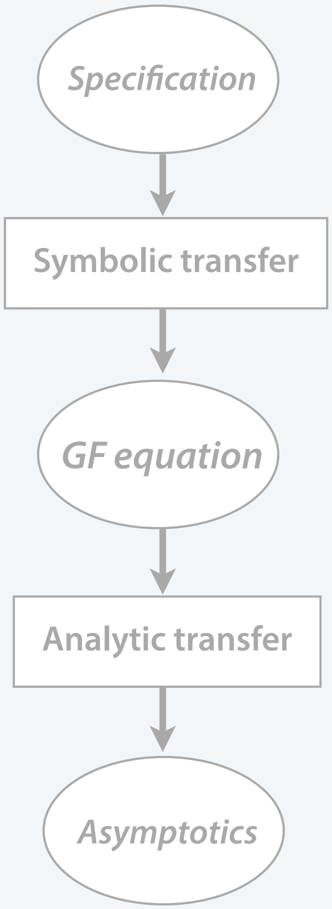
$$I_N = 1 \text{ for } N > 0$$



Singularity: *pole* at 1

Residue: $h_{-1} = -\dfrac{f(1)}{g'(1)} = 1$

Example 5: Compositions



C, the class of all compositions

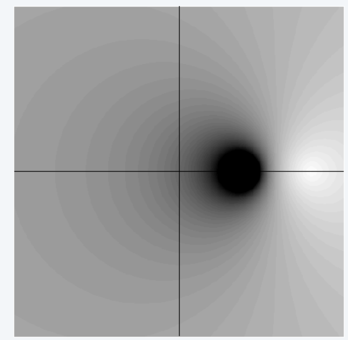
$$\mathcal{C} = \mathrm{SEQ}(\mathcal{I})$$

$$\downarrow$$

$$C(z) = \frac{1}{1 - I(z)} = \frac{1}{1 - \frac{z}{1-z}} = \frac{1 - z}{1 - 2z}$$

$$\downarrow$$

$$C_N = 2^{N-1} \quad \text{for } N > 0$$



Singularity: *pole* at 1/2

Residue: $h_{-1} = -\dfrac{f(1/2)}{g'(1/2)} = 1/4$

Example 5: Compositions

Q. How many ways to express N as a sum of 1s and 2s ?

$F_1 = 1$	$F_2 = 2$	$F_3 = 3$	$F_4 = 5$	$F_5 = 8$
1	1 + 1	1 + 1 + 1	1 + 1 + 1 + 1	1 + 1 + 1 + 1 + 1
	2	1 + 2	1 + 1 + 2	1 + 1 + 1 + 2
		2 + 1	1 + 2 + 1	1 + 1 + 2 + 1
			2 + 1 + 1	1 + 2 + 1 + 1
			2 + 2	1 + 2 + 2
				2 + 1 + 1 + 1
				2 + 1 + 2
				2 + 2 + 1

A. *Fibonacci numbers*

Example 5: Compositions

Specification

Symbolic transfer

GF equation

Analytic transfer

Asymptotics

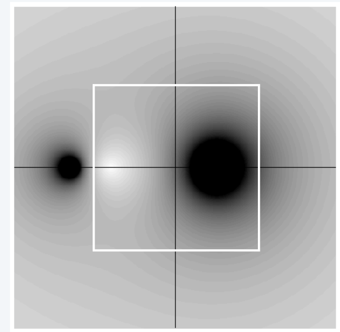
F, the class of all compositions composed of 1s and 2s

F = SEQ(**Z** + **Z**²)

$$F(z) = \frac{1}{1 - z - z^2}$$

$$F_N \sim \frac{\phi^N}{\sqrt{5}}$$

$$\frac{1}{\sqrt{5}} \doteq .4472 \text{ and } \phi \doteq 1.618$$



$$\hat{\phi} = \frac{\sqrt{5}-1}{2}$$

$$\phi = \frac{\sqrt{5}+1}{2}$$

Dominant singularity: *pole* at $\hat{\phi}$

Residue: $h_{-1} = -\frac{f(\hat{\phi})}{g'(\hat{\phi})} = \frac{1}{1+2\hat{\phi}}$

Coefficient of z^N: $\sim \frac{h_{-1}}{\hat{\phi}}\left(\frac{1}{\hat{\phi}}\right)^{N+1} = \frac{1}{1+2\hat{\phi}}\phi^N$

$$\phi\hat{\phi} = 1$$

$$\phi^2 = \phi + 1$$

$$1 + 2\hat{\phi} = \sqrt{5}$$

Example 5: Compositions

Q. How many ways to express N as a sum of primes ?

2

$P_2 = 1$

3

$P_3 = 1$

2 + 2

$P_4 = 1$

2 + 3
3 + 2
5

$P_5 = 3$

2 + 2 + 2
3 + 3

$P_6 = 2$

2 + 2 + 3
2 + 3 + 2
3 + 2 + 2
5 + 2
2 + 5
7

$P_7 = 6$

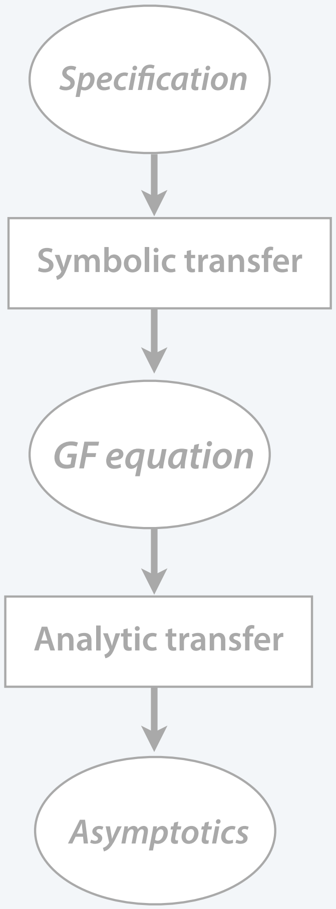
2 + 2 + 2 + 2
2 + 3 + 3
3 + 3 + 2
3 + 2 + 3
5 + 3
3 + 5

$P_8 = 6$

2 + 2 + 2 + 3
2 + 2 + 3 + 2
2 + 3 + 2 + 2
3 + 2 + 2 + 2
2 + 2 + 5
2 + 5 + 2
5 + 2 + 2
3 + 3 + 3
2 + 7
7 + 2

$P_9 = 10$

Example 5: Compositions

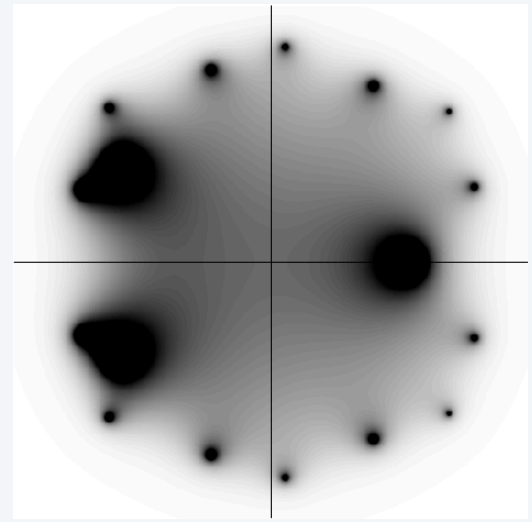


P, the class of all compositions
composed of primes

P = SEQ(**Z**^{2} + **Z**^{3} + **Z**^{5} + **Z**^{7} + . . .)

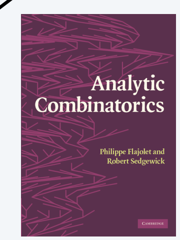
$$P(z) = \frac{1}{1 - z^2 - z^3 - z^5 - z^7 - z^{11} - \ldots}$$

$$[z^N]P(z) \sim \lambda\beta^N \quad \text{with} \begin{cases} \beta \doteq 1.4762 \\ \lambda \doteq .3037 \end{cases}$$



Dominant singularity: *pole* at $1/\beta \doteq .6774$

interesting calculations omitted (see text)

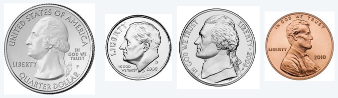


pp. 298–299

Note: periodic oscillations are present in the next term

Example 6: Denumerants (partitions from a fixed set)

Q. How many ways to make change for N cents?



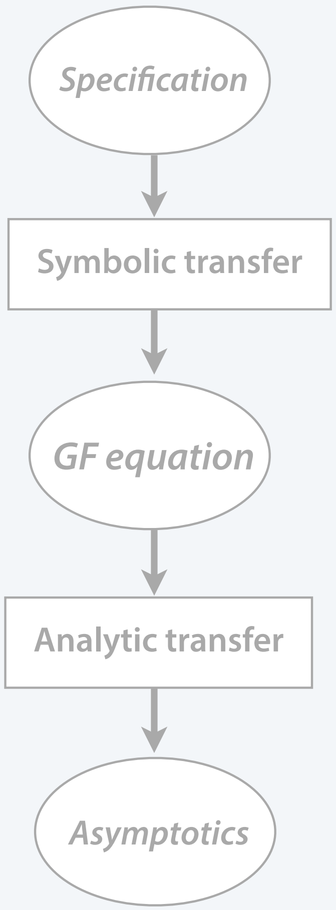
1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1
5 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1
5 + 5+ 1 + 1 + 1 + 1
10 + 1 + 1 + 1 + 1

$$Q_{14} = 4$$

1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1
5 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1
5 + 5+ 1 + 1 + 1 + 1 + 1
5 + 5 + 5
10 + 1 + 1 + 1 + 1 + 1
10 + 5

$$Q_{15} = 6$$

Example 6: Denumerants (partitions from a fixed set)

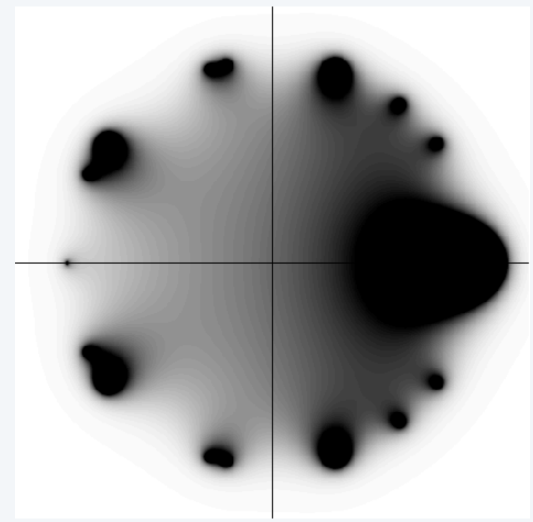


Q, the class of all partitions
composed of 1*s*, 5*s*, 10*s*, 25*s*

$\boldsymbol{Q} = \text{MSET}(\boldsymbol{Z} + \boldsymbol{Z}^{5} + \boldsymbol{Z}^{10} + \boldsymbol{Z}^{25})$

$$Q(z) = \frac{1}{(1-z)(1-z^5)(1-z^{10})(1-z^{25})}$$

$$[z^N]Q(z) \sim \frac{N^3}{1\cdot 5\cdot 10\cdot 25\cdot 3!} = \frac{N^3}{7500}$$

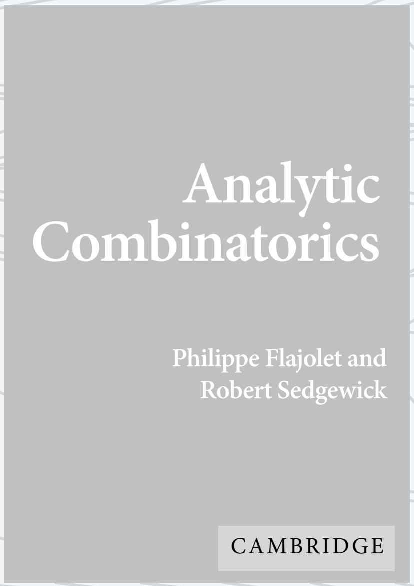


Dominant singularity: *pole of order* 5 *at* 1

Residue: $h_{-4} = \lim_{z\to 1}(1-z)^4 Q(z)$

$$= \frac{1}{1\cdot 5\cdot 10\cdot 25}$$

$$\lim_{z\to 1}\frac{1-z}{1-z^t} = \lim_{z\to 1}\frac{1}{1+z+z^2+\ldots+z^{t-1}} = \frac{1}{t}$$



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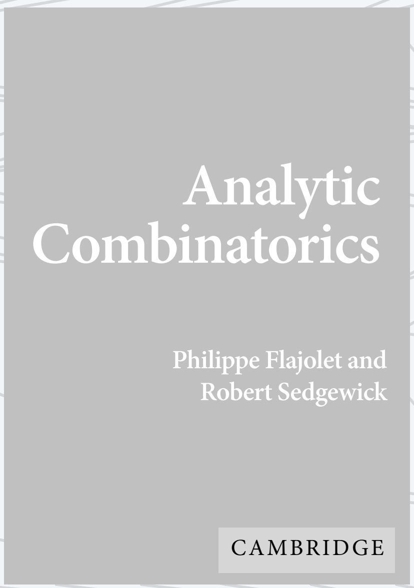
5. Applications of Rational and Meromorphic Asymptotics

- Bitstrings
- Other familiar examples
- **Compositions**
- Supercritical sequence schema



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5. Applications of Rational and Meromorphic Asymptotics

- Bitstrings
- Other familiar examples
- Restricted compositions
- **Supercritical sequence schema**

Sequence schema

Terminology. A *schema* is a treatment that unifies the analysis of a family of classes.

Definition. A class that admits a construction of the form **F** = *SEQ*(**G**), where **G** is any class (labelled or unlabelled) is said to be a *sequence class*, which falls within the *sequence schema*.

Enumeration:

F = *SEQ*(**G**) $\longrightarrow$ $F(z) = \dfrac{1}{1 - G(z)}$

$$f_N = [z^N]F(z)$$
$$g_N = [z^N]G(z)$$

unlabelled case: number of structures is f_N

labelled case: number of structures is $N!\,f_N$

Parameters:

mark number of **G** components with u

F = *SEQ*(u **G**) $\longrightarrow$ $F(z,u) = \dfrac{1}{1 - uG(z)}$

mark number of $\mathbf{G_k}$ components with u

F = *SEQ*(u $\mathbf{G_k}$ + $\mathbf{G} \backslash \mathbf{G_k}$) $\longrightarrow$ $F^k(z,u) = \dfrac{1}{1 - (G(z) + (u-1)g_k z^k)}$

Supercritical sequence classes

Supercriticality: A technical condition that enables us to unify the analysis of sequence classes.

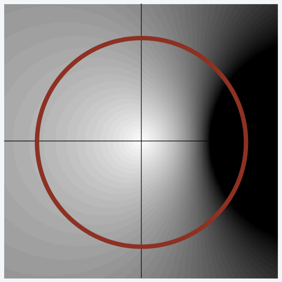
Definition. *Supercritical sequence classes.*
A sequence class **F** = SEQ(**G**) is said to be *supercritical* if $G(\rho) > 1$ where $G(z)$ is the generating function associated with **G** and $\rho>0$ is the radius of convergence of $G(z)$.

Example: GF for integers: $I(z) = \dfrac{z}{1-z}$

radius of convergence: $\rho = 1 - \epsilon$ for any $\epsilon > 0$

supercriticality test: $I(1-\epsilon) = \dfrac{1}{\epsilon} - 1 > 1$ for $\epsilon < 1/2$

Therefore, the class of compositions **C** = SEQ(**I**) is supercritical.



Note: For simplicity, we ignore periodicities in GFs in this lecture:

Definition. *Strong aperidoicity*. A GF $G(z)$ is said to be *strongly aperiodic* when there does not exist an integer $d > 1$ such that $G(z) = h(z^d)$ for some $h(z)$ analytic at 0.

Transfer theorem for supercritical sequence classes

Theorem. *Asymptotics of supercritical sequences.* If **F** = SEQ(**G**) is a strongly aperiodic supercritical sequence class, then $[z^N]F(z) \sim \dfrac{1}{G'(\lambda)} \dfrac{1}{\lambda^{N+1}}$ where λ is the root of $G(\lambda) = 1$ in (0, ρ).

radius of convergence of $G(z)$

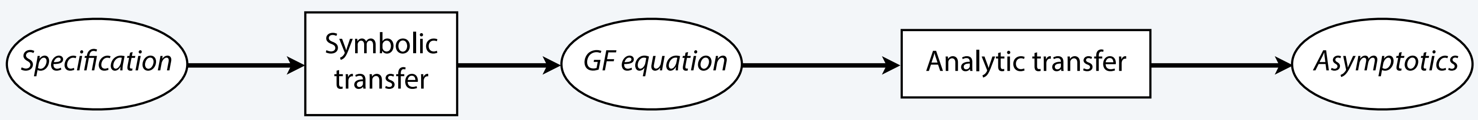
Proof sketch:

- $G(z)$ increases from $G(0) = 0$ to $G(\rho)>1$, so λ is well defined.
- At λ, $G(z)$ admits the series expansion $G(z) = 1 + G'(\lambda)(z-\lambda) + G''(\lambda)(z-\lambda)^2/2! + \cdots$
- Therefore, $F(z) = 1/(1-G(z))$ has a simple pole at λ, and $F(z) \sim -\dfrac{1}{G'(\lambda)(z-\lambda)} = \dfrac{1}{\lambda G'(\lambda)} \dfrac{1}{1-z/\lambda}$

Transfer theorem for supercritical sequence classes

Theorem. *Asymptotics of supercritical sequences.* If **F** = SEQ(**G**) is a strongly aperiodic supercritical sequence class, then $[z^N]F(z) \sim \frac{1}{G'(\lambda)}\frac{1}{\lambda^{N+1}}$ where λ is the root of $G(\lambda) = 1$ in (0, ρ).

	construction	$F(z)$	$G(z)$	λ	coefficient asymptotics
surjections	**R** = *SEQ* (*SET*$_{>0}$(**Z**))	$\frac{1}{2-e^z}$	$e^z - 1$	$\ln 2$	$\frac{N!}{2(\ln 2)^{N+1}}$
alignments	**O** = *SEQ* (*CYC*(**Z**))	$\frac{1}{1-\ln\frac{1}{1-z}}$	$\ln\frac{1}{1-z}$	$1-\frac{1}{e}$	$\frac{N!}{e(1-1/e)^{N+1}}$
compositions	**C** = *SEQ*(**I**)	$\frac{1}{1-\frac{z}{1-z}}$	$\frac{z}{1-z}$	$\frac{1}{2}$	2^{N-1}



Parts in compositions

Q. How many parts in a *random composition* of size *N* ?

1	1 + 1 2	1 + 1 + 1 1 + 2 2 + 1 3	1 + 1 + 1 + 1 1 + 1 + 2 1 + 2 + 1 1 + 3 2 + 1 + 1 2 + 2 3 + 1 4	1 + 1 + 1 + 1 + 1 1 + 1 + 1 + 2 1 + 1 + 2 + 1 1 + 1 + 3 1 + 2 + 1 + 1 1 + 2 + 2 1 + 3 + 1 1 + 4 2 + 1 + 1 + 1 2 + 1 + 2 2 + 2 + 1 2 + 3 3 + 1 + 1 3 + 2 4 + 1 5
1	1.5	2	2.5	3

Components in surjections

What is the expected value of M in a *random surjection* of size N?

1

1

1 1
1 2
2 1

$(1 + 2\cdot 2)/3 \doteq 1.666$

1 1 1	1 1 2	1 2 3
	1 2 1	1 3 2
	1 2 2	2 1 3
	2 1 1	2 3 1
	2 1 2	3 1 2
	2 2 1	3 2 1

$(1 + 2\cdot 6 + 3\cdot 6)/13 \doteq 2.384$

1 1 1 1	1 2 3 1	1 2 3 3	1 2 3 4
	1 3 2 1	1 3 2 3	1 3 2 4
1 1 1 2	2 1 3 1	2 1 3 3	2 1 3 4
1 1 2 1	2 3 1 1	2 3 1 3	2 3 1 4
1 2 1 1	3 1 2 1	3 1 2 3	3 1 2 4
2 1 1 1	3 2 1 1	3 2 1 3	3 2 1 4
1 1 2 2	1 2 1 3	1 3 3 2	1 2 4 3
1 2 1 2	1 3 1 2	2 3 3 1	1 3 4 2
2 1 1 2	2 1 1 3	3 1 3 2	2 1 4 3
2 1 2 1	3 1 1 2	3 2 3 1	2 3 4 1
2 2 1 1	1 1 2 3	3 3 1 2	3 1 4 2
1 2 2 1	1 1 3 2	3 3 2 1	3 2 4 1
1 2 2 2		1 2 3 2	1 4 2 3
2 1 2 2		1 3 2 2	1 4 3 2
2 2 1 2		2 1 3 2	2 4 1 3
2 2 2 1		2 3 1 2	2 4 3 1
		3 1 2 2	3 4 1 2
		3 2 1 2	3 4 2 1
		1 2 2 3	4 1 2 3
		2 1 2 3	4 1 3 2
		2 3 2 1	4 2 1 3
		3 2 2 1	4 2 3 1
		2 2 1 3	4 3 1 2
		2 2 3 1	4 3 2 1

$(1 + 2\cdot 14 + 3\cdot 36 + 4\cdot 24)/75 \doteq 3.106$

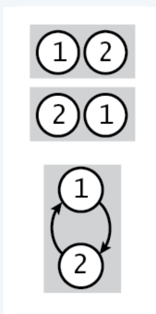
"coupon collector sequences"

For some M, each of the first M letters appears at least once.

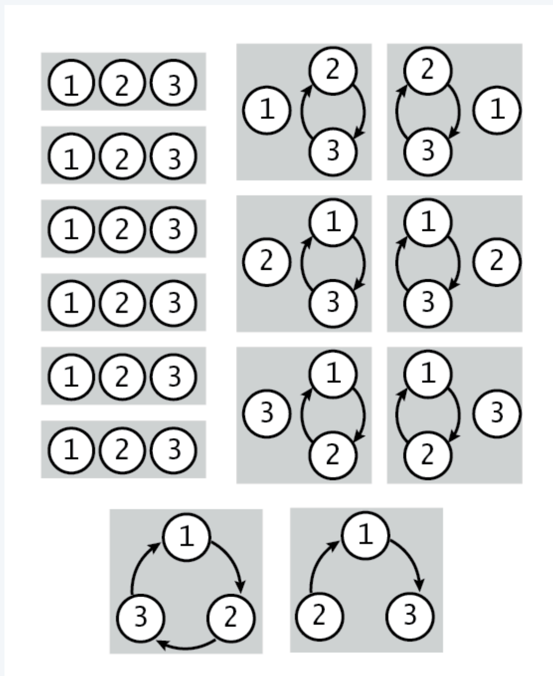
Components in alignments

How many cycles in a *random alignment* of size N?

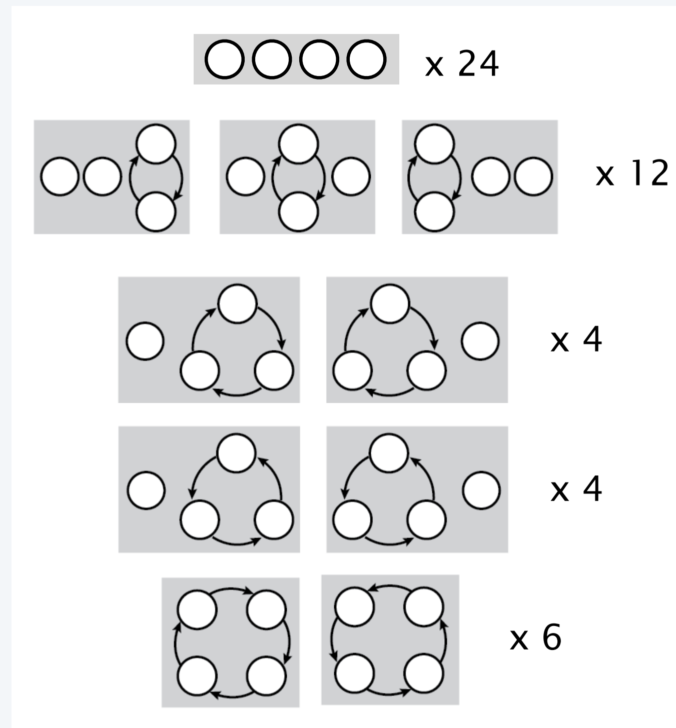
1



$(1 + 2 \cdot 2)/3 \doteq 1.666$

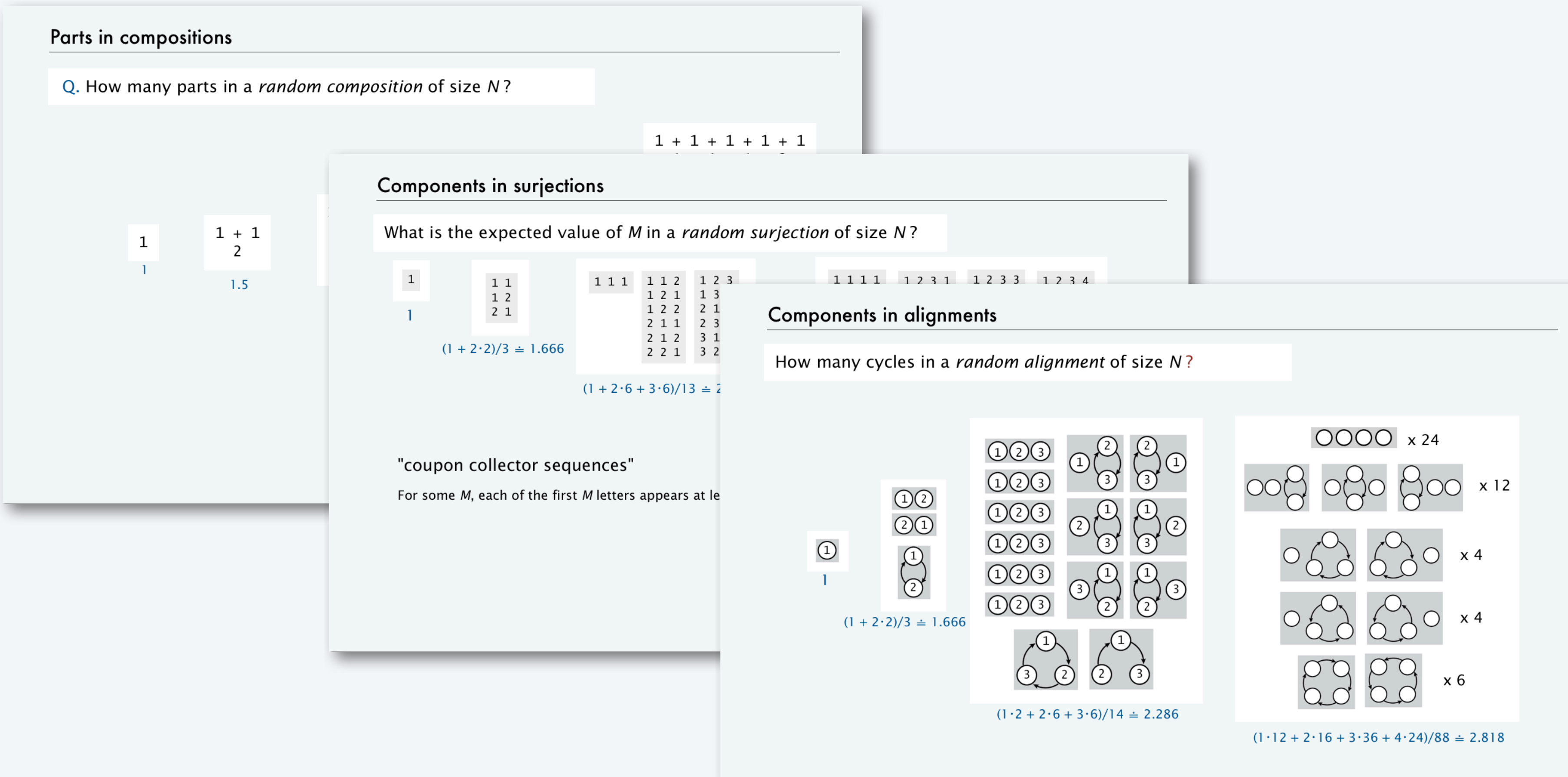


$(1 \cdot 2 + 2 \cdot 6 + 3 \cdot 6)/14 \doteq 2.286$



$(1 \cdot 12 + 2 \cdot 16 + 3 \cdot 36 + 4 \cdot 24)/88 \doteq 2.818$

A poster child for analytic combinatorics



Such questions can be answered *immediately* via *general transfer theorems*

Number of components in supercritical sequence classes

Corollary. *Number of components in supercritical sequence classes.* If **F** = SEQ(**G**) is a strongly aperiodic supercritical sequence class, then the expected number of *G*-components in a random *F*-component of size *N* is $\mu_N \sim \frac{N+1}{\lambda G'(\lambda)} + \frac{G''(\lambda)}{G'(\lambda)^2} - 1$ with variance $\sigma_N^2 \sim \frac{\lambda G''(\lambda) + G'(\lambda) - G'(\lambda)^2}{\lambda^2 G'(\lambda)^3} N$.

λ is the root of $G(\lambda) = 1$ in $(0, \rho)$

Proof idea:

$$\mu_N = \frac{1}{f_N}[z^N]\frac{\partial}{\partial u}\frac{1}{1 - uG(z)}\Big|_{u=1} = \frac{1}{f_N}[z^N]\frac{G(z)}{(1 - G(z))^2}$$

[further details omitted]

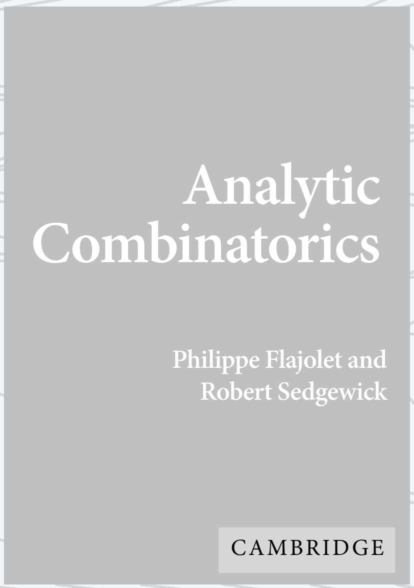
Number of components in supercritical sequence classes

Corollary. *Number of components in supercritical sequence classes.* If **F** = SEQ(**G**) is a strongly aperiodic supercritical sequence class, then the expected number of G-components in a random F-component of size N is $\mu_N \sim \dfrac{N+1}{\lambda G'(\lambda)} + \dfrac{G''(\lambda)}{G'(\lambda)^2} - 1$ with variance $\sigma_N^2 \sim \dfrac{\lambda G''(\lambda) + G'(\lambda) - G'(\lambda)^2}{\lambda^2 G'(\lambda)^3} N$.

λ is the root of $G(\lambda) = 1$ in $(0, \rho)$

	construction	$F(z)$	$G(z)$	λ	expected number of components
compositions	**C** = $SEQ($ **I** $)$	$\dfrac{1}{1-\frac{z}{1-z}}$	$\dfrac{z}{1-z}$	$\dfrac{1}{2}$	$\sim \dfrac{N}{2}$
surjections	**R** = $SEQ\,(SET_{>0}($ **Z** $))$	$\dfrac{1}{2-e^z}$	$e^z - 1$	$\ln 2$	$\sim \dfrac{N}{2\ln 2}$
alignments	**O** = $SEQ\,(CYC($ **Z** $))$	$\dfrac{1}{1-\ln\frac{1}{1-z}}$	$\ln\dfrac{1}{1-z}$	$1-\dfrac{1}{e}$	$\sim \dfrac{N}{e-1}$

Same idea extends to give profile of component sizes.



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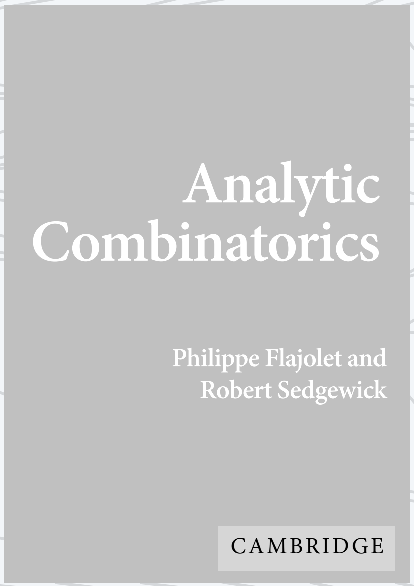
5. Applications of Rational and Meromorphic Asymptotics

- Bitstrings
- Other familiar examples
- Restricted compositions
- Supercritical sequence schema



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5. Applications of Rational and Meromorphic Asymptotics

- Bitstrings
- Other familiar examples
- Restricted compositions
- Supercritical sequence schema
- **Summary**

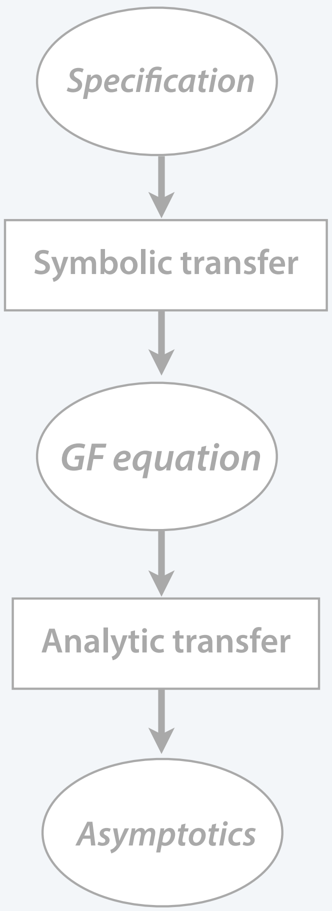
AC via meromophic asymptotics: summary of classic applications

class	*specification*	*generating function*	*coefficient asymptotics*
bitstrings	**B** = **E** + (**Z**$_0$ + **Z**$_1$) × **B**	$\frac{1}{1-2z}$	2^N
derangements	**D** = SET(CYC$_{>0}$(**Z**))	$\frac{e^{-z}}{1-z}$	$\sim \frac{N!}{e}$
surjections	**R** = SEQ(SET$_{>0}$(**Z**))	$\frac{1}{2-e^z}$	$\sim \frac{1}{2(\ln 2)^{N+1}}$
alignments	**O** = SEQ (*CYC*(**Z**))	$\frac{1}{1-\ln\frac{1}{1-z}}$	$\sim \frac{N!}{e(1-1/e)^{N+1}}$
set partitions	**S**$_r$= **Z**×SEQ(**Z**)×**Z**×SEQ(**Z**+**Z**)×...	$\frac{z^r}{(1-z)\ldots(1-rz)}$	$\sim \frac{r^N}{r!}$
integers	**I** = SEQ$_{>0}$(**Z**))	$\frac{z}{1-z}$	1
compositions	**C** = SEQ(**I**)	$\frac{1}{1-\frac{z}{1-z}}$	2^{N-1}

AC via meromophic asymptotics: summary of classic applications (variants)

class	*specification*	*generating function*	*coefficient asymptotics*
bitstrings with no 0000	$\mathbf{B}_4 = \mathbf{Z}_{<4}\ (\mathbf{E} + \mathbf{Z}_1\mathbf{B}_4)$	$\dfrac{1+z+z^2+z^3}{1-z-z^2-z^3-z^4}$	$1.092(1.928)^N$
generalized derangements	$\mathbf{D} = \mathrm{SET}(\mathrm{CYC}_{>M}(\ \mathbf{Z}\ ))$	$\dfrac{e^{-z-\frac{z^2}{2}-\frac{z^3}{3}-\ldots\ \frac{z^M}{M}}}{1-z}$	$\dfrac{N!}{e^{H_M}}$
double surjections	$\mathbf{R} = \mathrm{SEQ}(\mathrm{SET}_{>1}(\mathbf{Z}))$	$\dfrac{1}{2+z-e^z}$	$.4065\dfrac{N!}{(1.146)^N}$
compositions of 1s and 2s	$\mathbf{F} = \mathrm{SEQ}(\mathbf{Z} + \mathbf{Z}^2)$	$\dfrac{1}{1-z-z^2}$	$.4472(1.618)^N$
compositions of primes	$\mathbf{P} = \mathrm{SEQ}(\mathbf{Z}^2 + \mathbf{Z}^3 + \mathbf{Z}^5 + \ldots)$	$\dfrac{1}{1-z^2-z^3-z^5-z^7-\ldots}$	$.3037(1.476)^N$
denumerants	$\mathbf{Q} = \mathrm{MSET}(\mathbf{Z} + \mathbf{Z}^5 + \mathbf{Z}^{10} + \mathbf{Z}^{25})$	$\dfrac{1}{(1-z)(1-z^5)(1-z^{10})(1-z^{25})}$	$\dfrac{N^3}{7500}$

"If you can specify it, you can analyze it"



1. The transfer theorem for meromorphic GFs enables immediate analysis of a variety of classes.

class	*specification*	*generating function*	*coefficient asymptotics*
bitstrings	$\mathbf{B} = \mathbf{E} + (\mathbf{Z}_0 + \mathbf{Z}_1) \times \mathbf{B}$	$\frac{1}{1-2z}$	2^N
derangements	$\mathbf{D} = \mathrm{SET}(\mathrm{CYC}_{>0}(\mathbf{Z}))$	$\frac{e^{-z}}{1-z}$	$\frac{N!}{e}$
surjections	$\mathbf{R} = \mathrm{SEQ}(\mathrm{SET}_{>0}(\mathbf{Z}))$	$\frac{1}{2-e^z}$	$\frac{1}{2(\ln 2)^{N+1}}$
alignments	$\mathbf{O} = \mathrm{SEQ}(\mathit{CYC}(\mathbf{Z}))$	$\frac{1}{1-\ln\frac{1}{1-z}}$	$\frac{N!}{e(1-1/e)^{N+1}}$

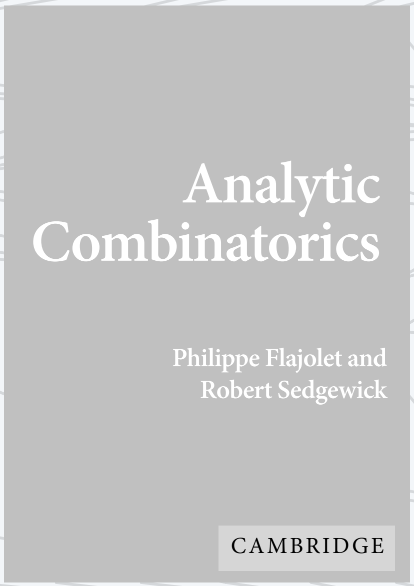
2. Variations are handled just as easily.

class	*specification*	*generating function*	*coefficient asymptotics*
bitstrings with no 0000	$\mathbf{B}_4 = \mathbf{Z}_{<4}(\mathbf{E} + \mathbf{Z}_1\mathbf{B}_4)$	$\frac{1+z+z^2+z^3}{1-z-z^2-z^3-z^4}$	$1.092(1.928)^N$
generalized derangements	$\mathbf{D} = \mathrm{SET}(\mathrm{CYC}_{>M}(\mathbf{Z}))$	$\frac{e^{-z-\frac{z^2}{2}-\frac{z^3}{3}-\ldots\frac{z^M}{M}}}{1-z}$	$\frac{N!}{e^{H_M}}$
double surjections	$\mathbf{R} = \mathrm{SEQ}(\mathrm{SET}_{>1}(\mathbf{Z}))$	$\frac{1}{2+z-e^z}$	$.4065\frac{N!}{(1.146)^N}$
compositions of 1s and 2s	$\mathbf{F} = \mathrm{SEQ}(\mathbf{Z} + \mathbf{Z}^2)$	$\frac{1}{1-z-z^2}$	$.4472(1.618)^N$
compositions of primes	$\mathbf{P} = \mathrm{SEQ}(\mathbf{Z}^2 + \mathbf{Z}^3 + \mathbf{Z}^5 + \ldots)$	$\frac{1}{1-z^2-z^3-z^5-z^7-\ldots}$	$.3037(1.476)^N$
denumerants	$\mathbf{Q} = \mathrm{MSET}(\mathbf{Z} + \mathbf{Z}^5 + \mathbf{Z}^{10} + \mathbf{Z}^{25})$	$\frac{1}{(1-z)(1-z^5)(1-z^{10})(1-z^{25})}$	$\frac{N^3}{7500}$

3. The *supercritical sequence schema* unifies the analysis for an entire family of classes, including analysis of parameters.

Note: Several other schemas have been developed (see text).

Next: GFs that are not meromorphic (singularities are *not* poles).



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5. Applications of Rational and Meromorphic Asymptotics

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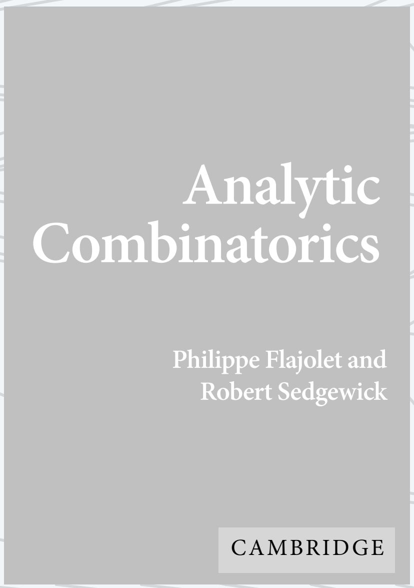
- **Summary**



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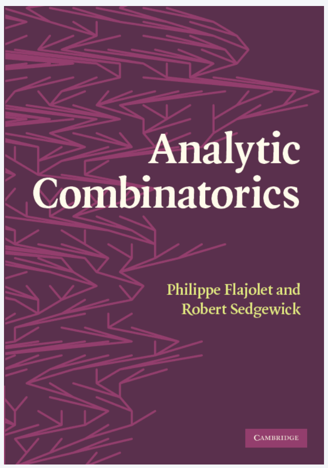
5. Applications of Rational and Meromorphic Asymptotics

- Bitstrings
- Other familiar examples
- Restricted compositions
- Supercritical sequence schema
- **Exercises**

Web Exercise V.1

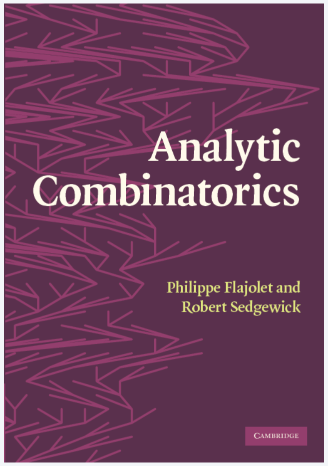
Patterns in strings.



Web Exercise V.1. Give an asymptotic expression for the number of strings that do not contain the pattern 0000000001. Do the same for 0101010101.

Web Exercise V.2

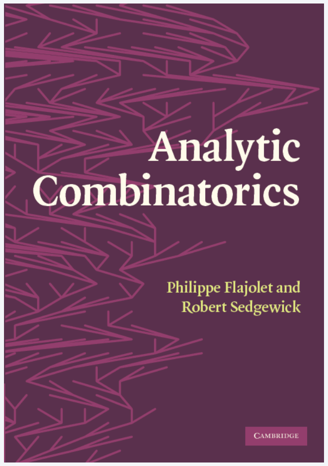
Variants of supercritical sequence classes.



Web Exercise V.2. Give asymptotic expressions for the number of objects of size N and the number of parts in a random object of size N for the following classes: compositions of 1s, 2s, and 3s, triple surjections, and alignments with no singleton cycles.

Assignments

1. Read pages 289-300 (*Applications of R&M Asymptotics*) in text. Skim pages 301-375.
   Usual caveat: Try to get a feeling for what's there, not understand every detail.



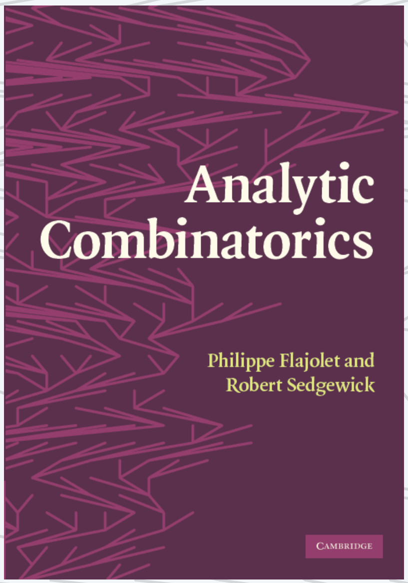
2. Write up solutions to Web exercises V.1 and V.2.

3. Programming exercise.

Program V.1. In the style of the plots in the lectures slides, plot the GFs for the set of bitstrings having no occurrence of the pattern 000000000. Do the same for 0101010101. (See Web Exercise V.1).

ANALYTIC COMBINATORICS

PART TWO



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5. Applications of Rational and Meromorphic Asymptotics