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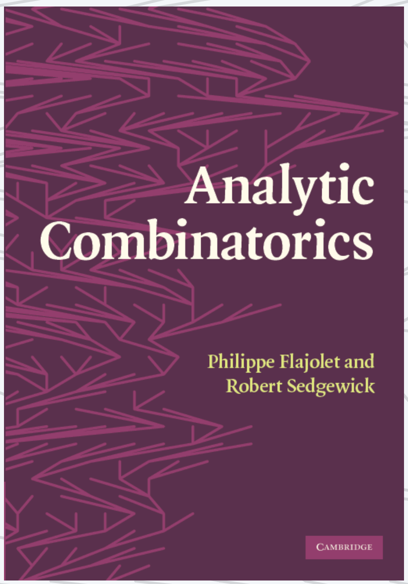

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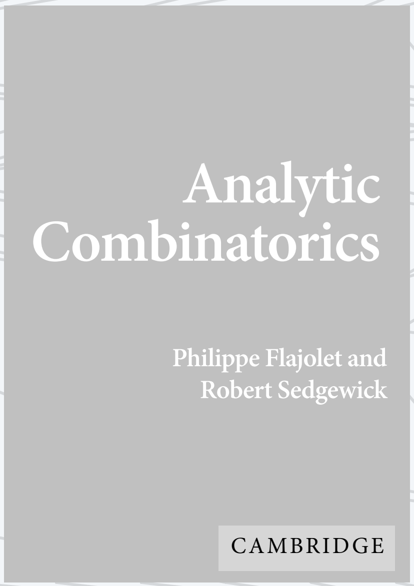
4. Complex Analysis, Rational and Meromorphic Asymptotics



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4. Complex Analysis, Rational and Meromorphic functions

- **Roadmap**
- Complex functions
- Rational functions
- Analytic functions and complex integration
- Meromorphic functions

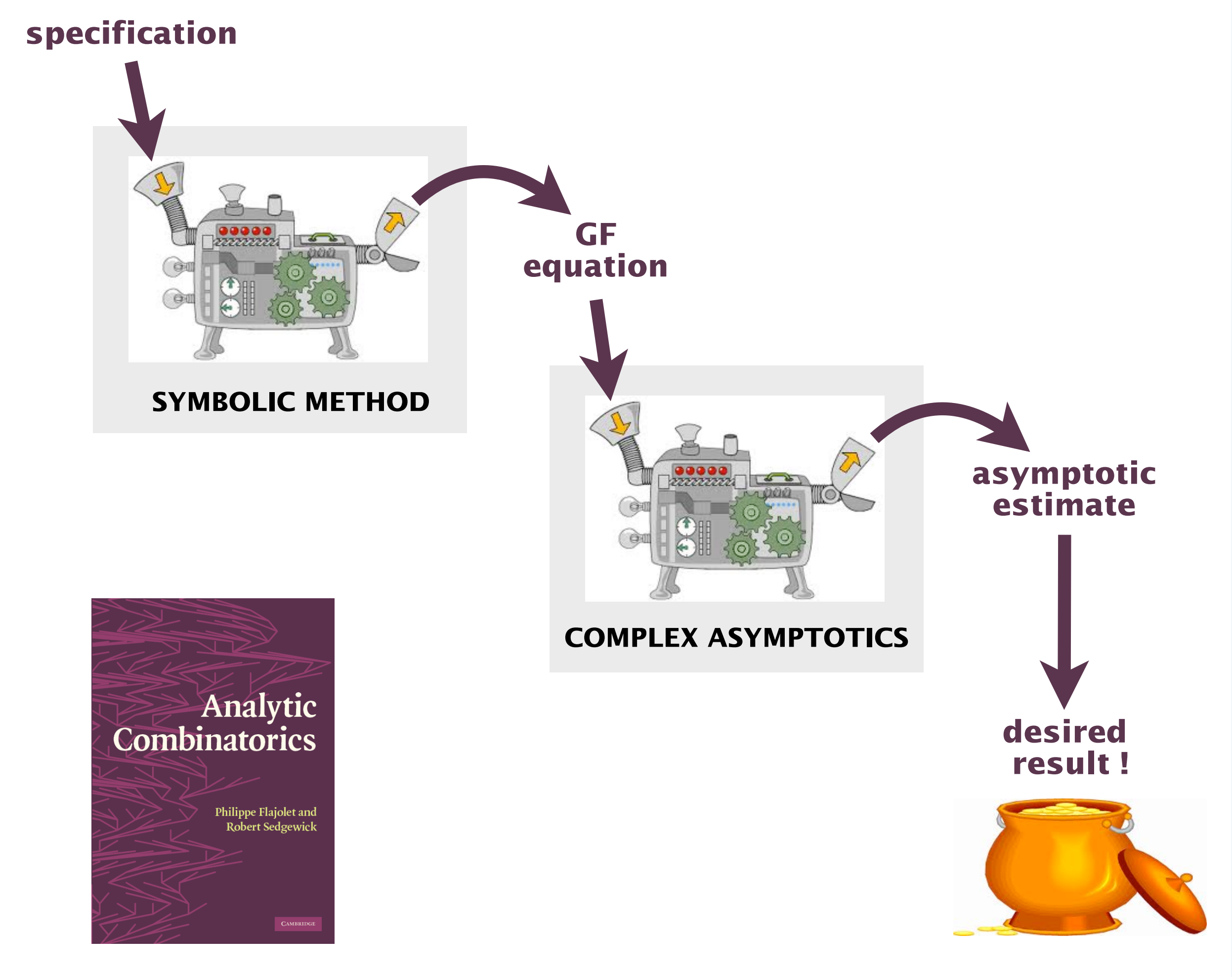
Analytic combinatorics overview

A. SYMBOLIC METHOD

1. OGFs
2. EGFs
3. MGFs

B. COMPLEX ASYMPTOTICS

4. Rational & Meromorphic
5. Applications of R&M
6. Singularity Analysis
7. Applications of SA
8. Saddle point



Starting point

The symbolic method supplies generating functions that vary widely in nature.

$$D(z) = \frac{e^{-z}}{1-z} \qquad G(z) = \frac{1+\sqrt{1-4z}}{2} \qquad R(z) = \frac{1}{2-e^z} \qquad B_P(z) = \frac{1+z+z^2+\ldots+z^{P-1}}{1-z-z^2-\ldots-z^P}$$

$$S_r(z) = \frac{z^r}{(1-z)(1-2z)\ldots(1-rz)} \qquad C(z) = \frac{1}{1-z}\ln\frac{1}{1-z} \qquad I(z) = e^{z+z^2/2}$$

Next step: Derive asymptotic estimates of coefficients.

$$[z^N]D(z) \sim \frac{1}{e} \qquad [z^N]G(z) \sim \frac{4^{N-1}}{\sqrt{\pi N^3}} \qquad [z^N]R(z) \sim \frac{1}{2(\ln 2)^{N+1}} \qquad [z^N]B_P(z) = C\beta^N$$

$$[z^N]S_r(z) \sim \frac{r^N}{r!} \qquad [z^N]C(z) = \ln N \qquad [z^N]I(z) \sim \frac{e^{N/2-\sqrt{N}-1/4}N^{-N/2}}{\sqrt{4\pi N}}$$

Classical approach: Develop explicit expressions for coefficients, then approximate

Analytic combinatorics approach: *Direct* approximations.

Starting point

Catalan trees

Construction	**G** = ○ × *SEQ*(**G**)
OGF equation	$G(z) = \frac{1}{1 - G(z)}$
Explicit form of OGF	$G(z) = \frac{1 + \sqrt{1-4z}}{2}$
Expansion	$G(z) = -\frac{1}{2}\sum_{N\geq 1}\binom{\frac{1}{2}}{N}(-4z)^N$
Explicit form of coefficients	$G_N = \frac{1}{N}\binom{2N-2}{N-1}$
Approximation	$G_N \sim \frac{4^{N-1}}{\sqrt{\pi N^3}}$

Derangements

Construction	**D** = *SET* (*CYC*$_{>1}$(**Z**))
EGF equation	$D(z) = e^{\ln\frac{1}{1-z} - z}$
Explicit form of EGF	$= \frac{e^{-z}}{1-z}$
Expansion	$D(z) = \Big(\sum_{k\geq 0}\frac{(-z)^k}{k!}\Big)\Big(\sum_{N\geq 0} z^N\Big)$
Explicit form of coefficients	$D_N = \sum_{0\leq k\leq N}\frac{(-1)^k}{k!}$
Approximation	$D_N \sim e^{-1}$

Problem: Explicit forms can be unwieldy (or unavailable).

$$\frac{e^{-z - z^2/2 - z^3/3}}{1-z}$$

$$(1 + z + z^2/2! + \ldots + z^b/b!)^M$$

Opportunity: Relationship between asymptotic result and GF.

Analytic combinatorics overview

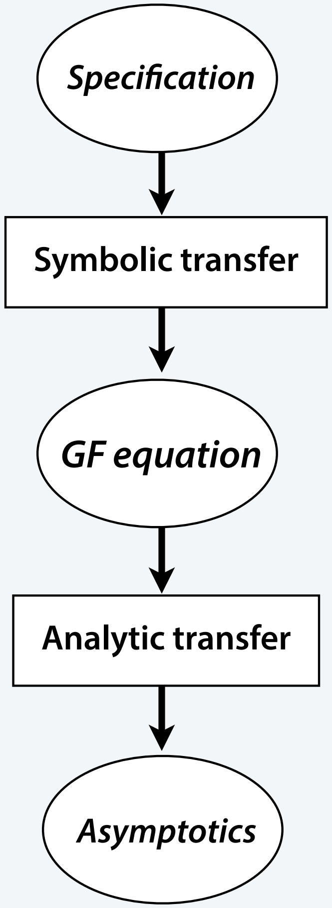
To analyze properties of a large combinatorial structure:

1. Use the symbolic method (lectures 1 and 2).
   - Define a *class* of combinatorial objects.
   - Define a notion of *size* (and associated GF)
   - Use standard constructions to *specify* the structure.
   - Use a *symbolic* transfer theorem.

Result: A direct derivation of a GF equation.

2. Use complex asymptotics (starting with this lecture).
   - Start with GF equation.
   - Use an *analytic* transfer theorem.

Result: Asymptotic estimates of the desired properties.



Ex. Derangements

$\mathbf{D} = SET\,(CYC_{>1}(\,\mathbf{Z}\,))$

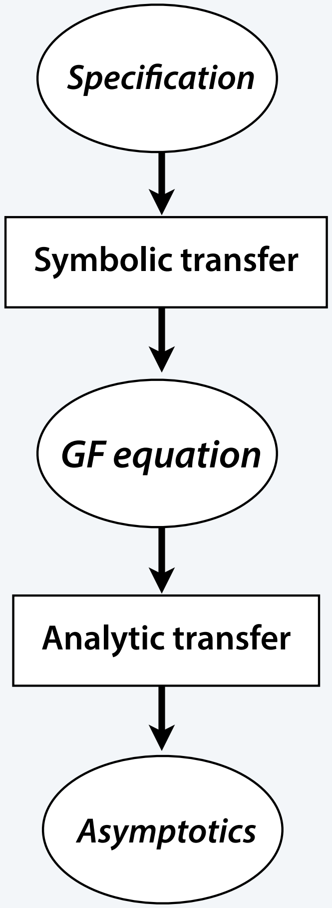
↓

$$D(z) = \frac{e^{-z}}{1-z}$$

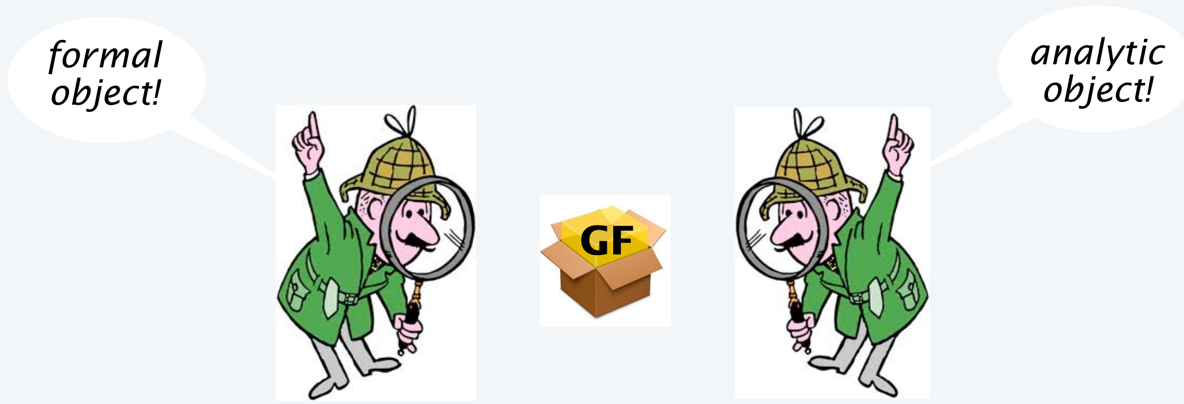
↓

$$D_N \sim \frac{1}{e}$$

A shift in point of view



generating functions are treated as *formal* objects

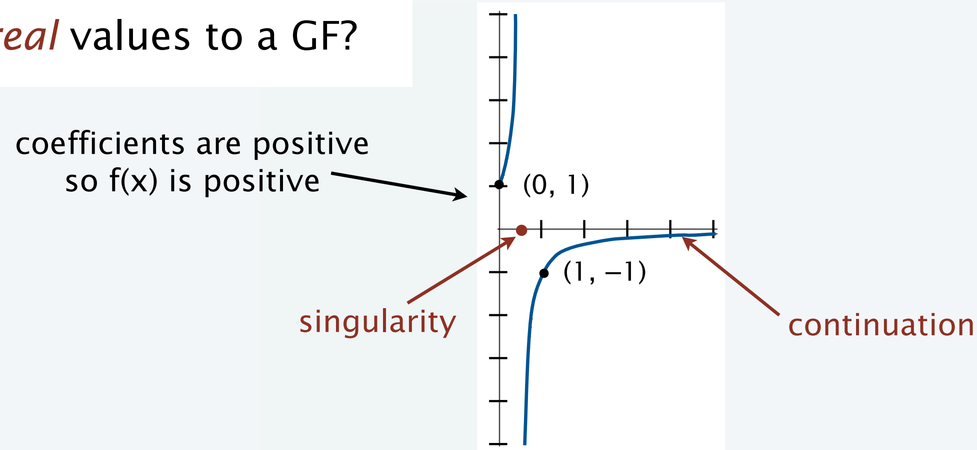


generating functions are treated as *analytic* objects

GFs as analytic objects (real)

Q. What happens when we assign *real* values to a GF?

$$f(x) = \frac{1}{1-2x}$$



A. We can use a *series representation* (in a certain interval) that allows us to extract coefficients.

$$\frac{1}{1-2x} = 1 + 2x + 4x^2 + 8x^3 + \ldots \qquad \text{for } 0 \le x < 1/2 \qquad [z^n]f(x) = 2^n$$

Useful concepts:

Differentiation: Compute derivative term-by-term where series is valid. $f'(x) = 2 + 8x + 24x^2 + \ldots$

Singularities: Points at which series ceases to be valid.

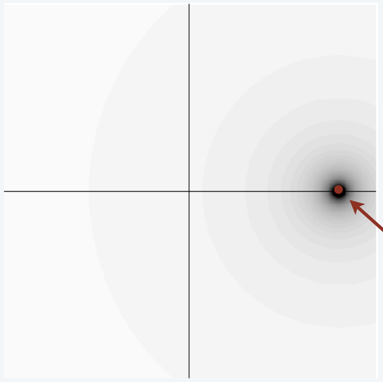
Continuation: Use functional representation even where series may diverge. $f(1) = -1$

GFs as analytic objects (complex)

Q. What happens when we assign *complex* values to a GF?

$$f(z) = \frac{e^{-z}}{1 - z}$$

stay tuned for interpretation of plot



A. We can use a *series representation* (in a certain domain) that allows us to extract coefficients.

Same useful concepts:

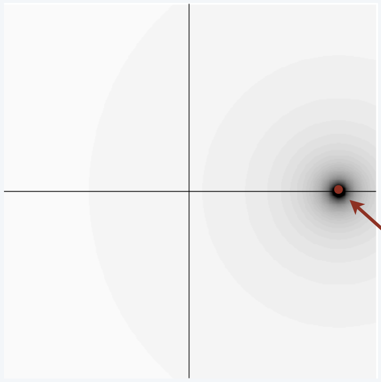
- Differentiation: Compute derivative term-by-term where series is valid.
- Singularities: Points at which series ceases to be valid.
- Continuation: Use functional representation even where series may diverge.

GFs as analytic objects (complex)

Q. What happens when we assign *complex* values to a GF?

$$f(z) = \frac{e^{-z}}{1-z}$$

stay tuned for interpretation of plot



A. A surprise!

Singularities provide *full information* on growth of GF coefficients!

"Singularities provide a royal road to coefficient asymptotics."

General form of coefficients of combinatorial GFs

$$[z^N]F(z) = A^N\theta(N)$$

exponential growth factor: A^N; subexponential factor: $\theta(N)$

First principle of coefficient asymptotics

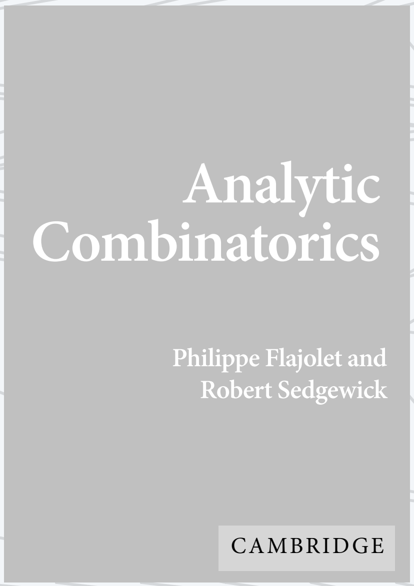
The *location* of a function's singularities dictates the *exponential growth* of its coefficients.

Second principle of coefficient asymptotics

The *nature* of a function's singularities dictates the *subexponential factor* of the growth.

Examples (preview):

	GF	GF type	singularities: location	singularities: nature	exponential growth	subexp. factor
strings with no 00	$B_2(z) = \dfrac{1-z^2}{1-2z-z^3}$	*rational*	$1/\phi, 1/\hat{\phi}$	*pole*	ϕ^N	$\dfrac{1}{\sqrt{5}}$
derangements	$D(z) = \dfrac{e^{-z}}{1-z}$	*meromorphic*	1	*pole*	1^N	e^{-1}
Catalan trees	$G(z) = \dfrac{1+\sqrt{1-4z}}{2}$	*analytic*	$1/4$	*square root*	4^N	$\dfrac{1}{4\sqrt{\pi N^3}}$



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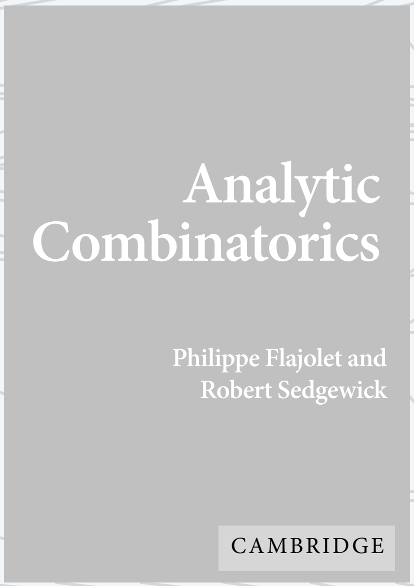

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Theory of complex functions

Quintessential example of the power of abstraction.

Start by defining i to be the square root of -1 so that $i^2 = -1$

Continue by exploring natural definitions of basic operations

- Addition
- Multiplication
- Division
- Exponentiation
- Functions
- Differentiation
- Integration

$1 + i$

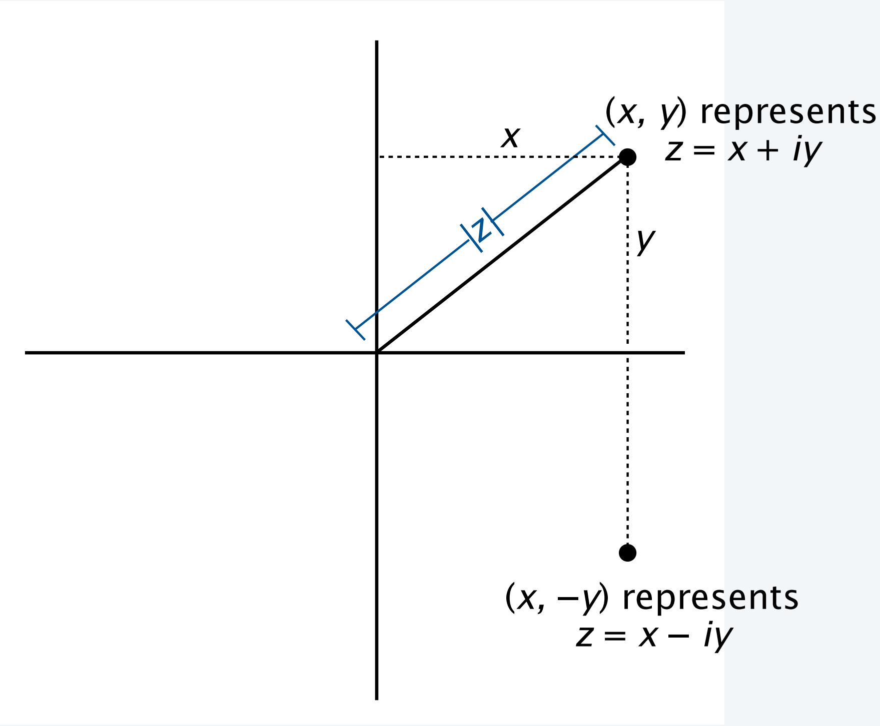
Standard conventions

$z = x + iy$

real part	$\Re z \equiv x$
imaginary part	$\Im z \equiv y$
absolute value	$\lvert z \rvert \equiv \sqrt{x^2 + y^2}$
conjugate	$\bar{z} = x - iy$

Quick exercise: $z\bar{z} = \lvert z \rvert^2$

Correspondence with points in the plane



Basic operations

Natural approach: Use algebra, but convert i^2 to -1 whenever it occurs

Addition

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

Multiplication

$$\begin{aligned}(a + bi) * (c + di) &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + (bc + ad)i\end{aligned}$$

Division

$$\frac{1}{a + bi} = \frac{a - bi}{a^2 + b^2} \qquad \frac{1}{z} = \frac{\bar{z}}{|z|}$$

Exponentiation?

Analytic functions

Definition. A function $f(z)$ defined in Ω is *analytic* at a point z_0 in Ω iff for z in an open disc in Ω centered at z_0 it is representable by a power-series expansion $f(z) = \sum_{N \geq 0} c_N (z - z_0)^N$

Examples:

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + z^4 + \ldots$$ is analytic for $|z| < 1$.

$$e^z \equiv 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \ldots$$ is analytic for $|z| < \infty$.

Complex differentiation

Definition. A function $f(z)$ defined in a region Ω is *holomorphic* or *complex-differentiable* at a point z_0 in Ω iff the limit $f'(z_0) = \lim_{\delta \to 0} \frac{f(z_0 + \delta) - f(z_0)}{\delta}$ exists, for complex δ.

Note: Notationally the same as for reals, but *much stronger*—the value is independent of the way that δ approaches 0.

Theorem. *Basic Equivalence Theorem.*
A function is *analytic* in a region Ω iff it is *complex-differentiable* in Ω.

For purposes of this lecture: Axiom 1.

Useful facts:

- If function is analytic (complex-differentiable) in Ω, it admits derivatives of any order in Ω.
- We can differentiate a function via term-by-term differentiation of its series representation.
- Taylor series expansions ala reals are effective.

Taylor's theorem

immediately gives power series expansions for analytic functions.

$$e^z \equiv 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \ldots$$

$$\sin z \equiv \frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots$$

$$\cos z \equiv 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots$$

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + z^4 + \ldots$$

Euler's formula

Evaluate the exponential function at $i\theta$

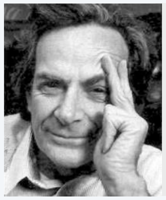
$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \ldots + \frac{(i\theta)^n}{n!} + \ldots$$

$$= 1 + i\frac{\theta}{1!} - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\frac{\theta^7}{7!} + \frac{\theta^8}{8!} + i\frac{\theta^9}{9!} + \ldots$$

$i^2 = -1$ $i^3 = -i$ $i^4 = 1$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Euler's formula



"Our jewel . . . one of the most remarkable, almost astounding, formulas in all of mathematics"

— Richard Feynman, 1977

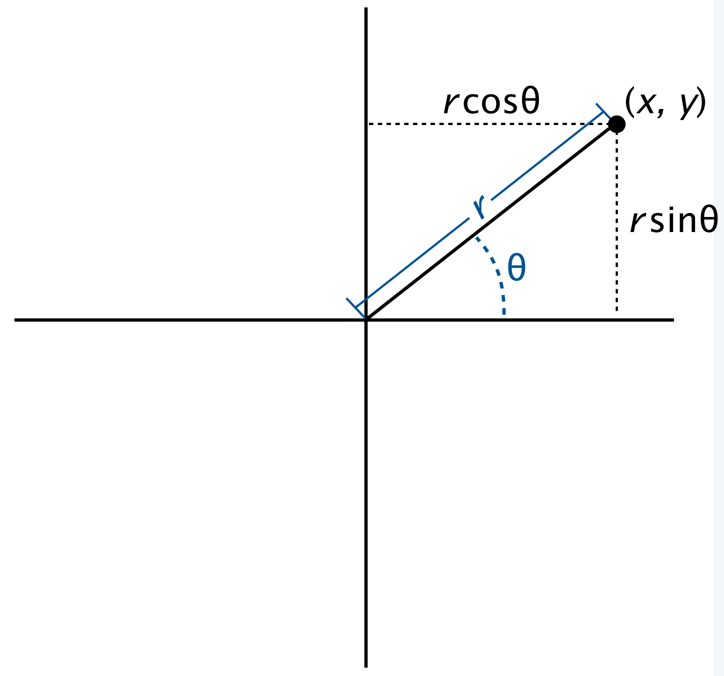
Polar coordinates

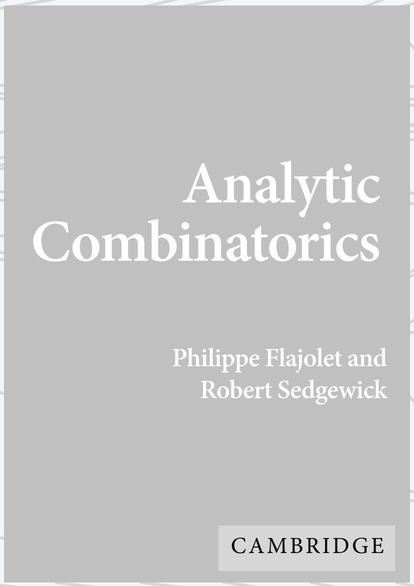
Euler's formula gives another correspondence between complex numbers and points in the plane.

$$re^{i\theta} = r\cos\theta + ir\sin\theta$$

Conversion functions defined for any complex number $x + iy$:

- absolute value (modulus) $\quad r = \sqrt{x^2 + y^2}$
- angle (argument) $\quad \theta = \arctan \frac{y}{x}$





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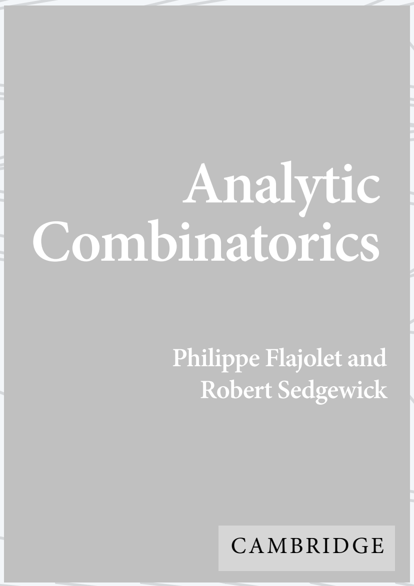

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Rational functions

are complex functions that are the ratio of two *polynomials*.

$$D(z) = \frac{e^{-z}}{1-z} \qquad G(z) = \frac{1+\sqrt{1-4z}}{2} \qquad R(z) = \frac{1}{2-e^z} \qquad B_P(z) = \frac{1+z+z^2+\ldots+z^{P-1}}{1-z-z^2-\ldots-z^P}$$

$$S_r(z) = \frac{z^r}{(1-z)(1-2z)\ldots(1-rz)} \qquad C(z) = \frac{1}{1-z}\ln\frac{1}{1-z} \qquad I(z) = e^{z+z^2/2}$$

Approach:

- Use *partial fractions* to expand into terms for which coefficient extraction is easy.
- Focus on the largest term to approximate.

[Same approach as for reals, but takes complex roots into account.]

Extracting coefficients from rational GFs

Factor the denominator and use *partial fractions* to expand into sum of simple terms.

$$A(z) \equiv \sum_{N\geq 0} a_N z^N$$

Example 1.
(distinct roots)

Rational GF	$A(z) = \dfrac{z}{1-5z+6z^2}$
Factor denominator	$= \dfrac{z}{(1-3z)(1-2z)}$
Use partial fractions: Expansion must be of the form	$A(z) = \dfrac{c_0}{1-3z} + \dfrac{c_1}{1-2z}$
Cross multiply and solve for coefficients.	$c_0 + c_1 = 0$ $2c_0 + 3c_1 = -1$
Solution is $c_0 = 1$ and $c_1 = -1$	$A(z) = \dfrac{1}{1-3z} - \dfrac{1}{1-2z}$
Extract coefficients.	$a_N = [z^N]A(z) = 3^N - 2^N$

Extracting coefficients from rational GFs

Factor the denominator and use *partial fractions* to expand into sum of simple terms.

Example 2.
(multiple roots)

Rational GF

$$A(z) = \frac{z^2}{1 - 3z + 4z^3}$$

Factor denominator

$$= \frac{z}{(1+z)(1-2z)^2}$$

Use partial fractions:
Expansion must be of the form

$$A(z) = \frac{c_0}{1+z} + \frac{c_1}{1-2z} + \frac{c_2}{(1-2z)^2}$$

Cross multiply
and solve for coefficients.

$$c_0 + c_1 + c_2 = 0$$
$$-4c_0 - c_1 + c_2 = 1$$
$$4c_0 - 2c_1 = 0$$

Solution is
$c_0 = -2/9$, $c_1 = -1/9$, and $c_2 = 3/9$

$$A(z) = \frac{1}{9}\left(-\frac{1}{1+z} - \frac{2}{1-2z} + \frac{3}{(1-2z)^2}\right)$$

Extract coefficients.

$$a_N = [z^N]A(z) = \frac{1}{9}(-(-1)^N + 2^N + 3N2^N)$$

Approximating coefficients from rational GFs

When roots are real, *only one term matters*.

$$A(z) = \frac{1}{9}\Big(-\frac{1}{1+z} - \frac{2}{1-2z} + \frac{3}{(1-2z)^2}\Big)$$

$$a_N = \frac{1}{9}(-(-1)^N + 2^N + 3N2^N)$$

$$a_N \sim \frac{1}{9}(2^N + 3N2^N)$$

smaller roots give exponentially smaller terms

$$a_N \sim \frac{1}{3}N2^N$$

multiplicity 3 gives terms of the form $n^2\beta^n$, etc.

Extracting coefficients from rational GFs

Factor the denominator and use *partial fractions* to expand into sum of simple terms.

Example 3.
(complex roots)

Rational GF

$$A(z) = \frac{1-2z}{1-2z+z^2-2z^3}$$

Factor denominator

$$= \frac{1-2z}{(1-2z)(1+z^2)} = \frac{1}{(1+z^2)}$$

Use partial fractions:
Expansion must be of the form

$$A(z) = \frac{c_0}{1-iz} + \frac{c_1}{1+iz}$$

Cross multiply
and solve for coefficients.

$$c_0 + c_1 = 1$$
$$ic_0 - ic_1 = 0$$

Solution is
$c_0 = c_1 = 1/2$

$$A(z) = \frac{1}{2}\Big(\frac{1}{1-iz} + \frac{1}{1+iz}\Big)$$

Extract coefficients.

$$[z^N]A(z) = \frac{1}{2}(i^N + (-i)^N) = \frac{1}{2}i^N(1+(-1)^N)$$

`1, 0, -1, 0, 1, 0, -1, 0, 1...`

Extracting coefficients form rational GFs (summary)

Theorem. Suppose that $g(z)$ is a polynomial of degree t with roots $\beta_1, \beta_2, \ldots, \beta_r$ and let m_i denote the multiplicity of β_i for i from 1 to r. If $f(z)$ is another polynomial with no roots in common with $g(z)$, and $g(0) \neq 0$ then

$$[z^N]\frac{f(z)}{g(z)} = \sum_{0 \le j < m_1} c_{1j} n^j \beta_1^n + \sum_{0 \le j < m_2} c_{2j} n^j \beta_2^n + \ldots + \sum_{0 \le j < m_r} c_{rj} n^j \beta_r^n$$

Notes:

- There are t terms, because $m_1 + m_2 + \ldots + m_r = t$.
- The t constants c_{ij} depend upon f.
- Complex roots introduce periodic behavior.

AC transfer theorem for rational GFs (leading term)

Theorem. Assume that a rational GF $f(z)/g(z)$ with $f(z)$ and $g(z)$ relatively prime and $g(0)\neq 0$ has a *unique pole of smallest modulus* $1/\beta$ and that the multiplicity of β *is* ν. Then

$$[z^n]\frac{f(z)}{g(z)} \sim C\beta^n n^{\nu-1} \quad \text{where} \quad C = \nu\frac{(-\beta)^\nu f(1/\beta)}{g^{(\nu)}(1/\beta)}$$

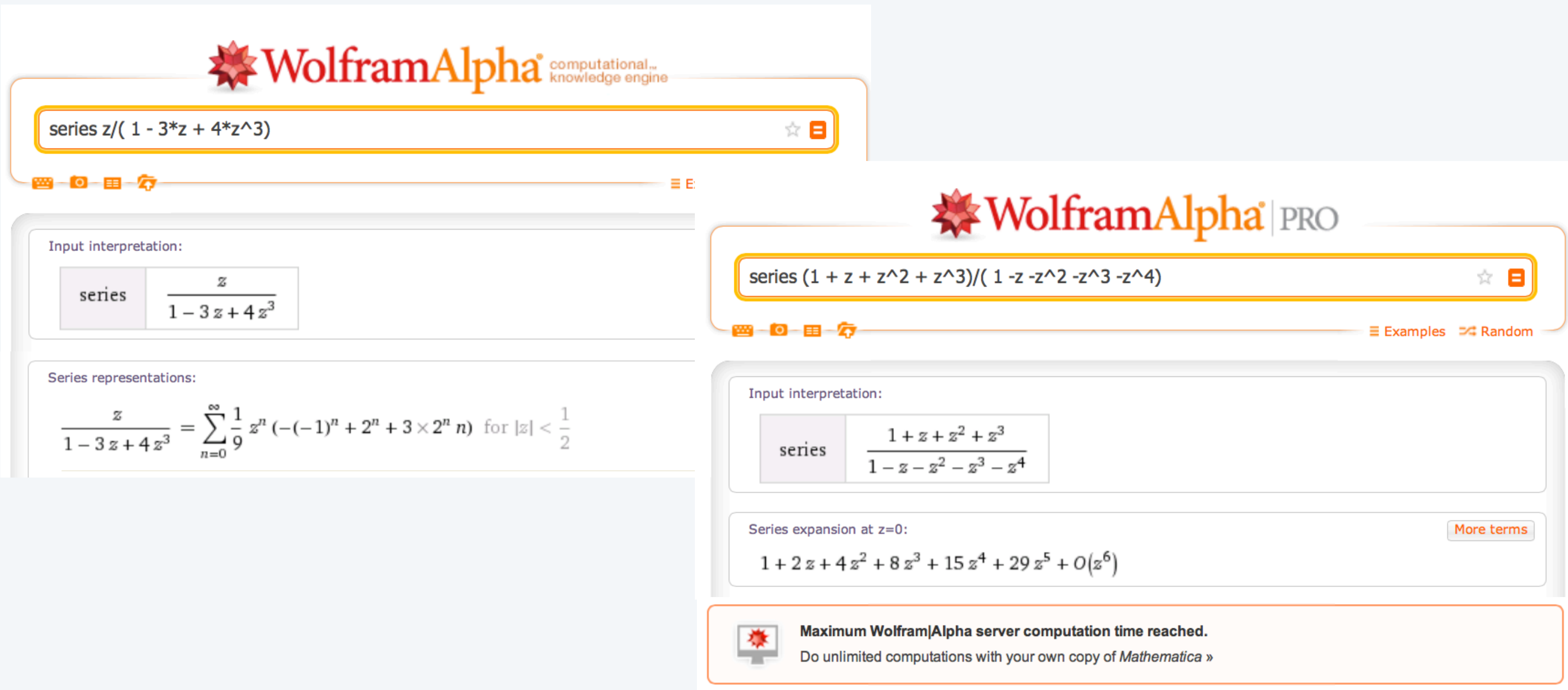
typical case

Examples.

$A(z) = f(z)/g(z)$	$1/\beta$	ν	C	$[z^N]A(z)$
$\frac{z}{1-3z+4z^3}$	1/2	2	$2\frac{(-2)^2(1/2)}{12} = \frac{1}{3}$	$\sim \frac{1}{3}N2^N$
$\frac{1}{1-z-z^2}$	φ	1	$\frac{(-1/\phi)}{-1-(2/\phi)} = \frac{1}{\sqrt{5}}$	$\sim \frac{1}{\sqrt{5}}\phi^N$
$\frac{1+z+z^2+z^3}{1-z-z^2-z^3-z^4}$	1.9276...	1	1.09166...	$\sim C\beta^N$

Computer algebra solution

Transfer theorem amounts to an *algorithm* that is embodied in many computer algebra systems.

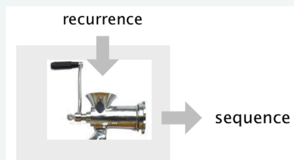


Classic example: Algorithm for solving linear recurrences

Solving recurrences with OGFs

General procedure:

- Make recurrence valid for all n.
- Multiply both sides of the recurrence by z^n and sum on n.
- Evaluate the sums to derive an equation satisfied by the OGF.
- Solve the equation to derive an explicit formula for the OGF. (Use the initial conditions!)
- Expand the OGF to find coefficients.



Asymptotics of linear recurrences

Theorem. Assume that a rational GF $f(z)/g(z)$ with $f(z)$ and $g(z)$ relatively prime and $g(0)=0$ has a unique pole $1/\beta$ of smallest modulus and that the multiplicity of β *is* ν. Then

$$[z^n]\frac{f(z)}{g(z)} \sim C\beta^n n^{\nu-1} \quad \text{where} \quad C = \nu\frac{(-\beta)^\nu f(1/\beta)}{g^{(\nu)}(1/\beta)}$$

Example from earlier lectures.

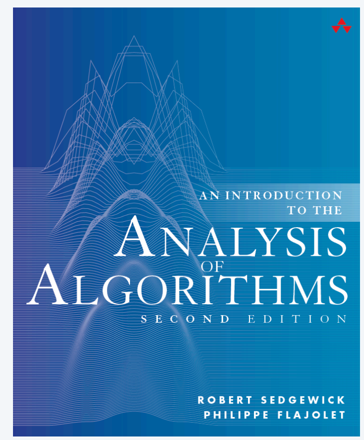
$$a_n = 5a_{n-1} - 6a_{n-2} \quad \text{for } n \geq 2 \text{ with } a_0 = 0 \text{ and } a_1 = 1$$

Make recurrence valid for all n. $\quad a_n = 5a_{n-1} - 6a_{n-2} + \delta_{n1}$

Multiply by z^n and sum on n. $\quad A(z) = 5zA(z) - 6z^2A(z) + z$

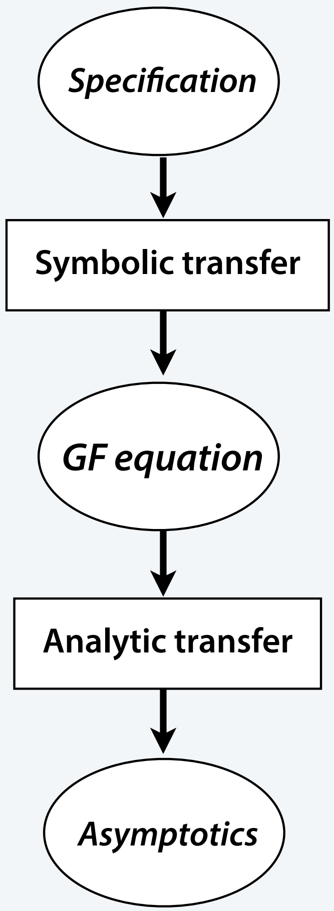
Solve. $\quad A(z) = \dfrac{z}{1 - 5z + 6z^2}$

Smallest root of denominator is 1/3. $\quad a_n \sim 3^n \quad C = 1\dfrac{(-3)(1/3)}{-5 + 12/3} = 1$



pp. 157–158

AC example with rational GFs: Patterns in strings



B_4, the class of all binary strings with no 0^4 ← see Lecture 1

$$\mathbf{B_4 = Z_{<4}\,(E + Z_1 B_4)}$$

↓

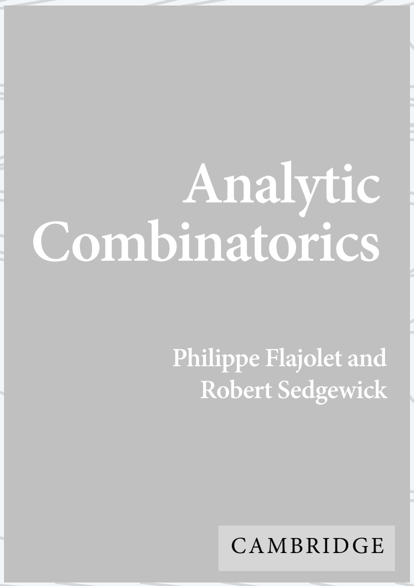
$$B_4(z) = (1 + z + z^2 + z^3)(1 + zB_4(z))$$

$$= \frac{1 + z + z^2 + z^3}{1 - z - z^2 - z^3 - z^4}$$

↓

$$B_{4N} \sim C\beta^N \quad \text{with } C \doteq 1.0917 \text{ and } \beta \doteq 1.9276$$

Many more examples to follow (next lecture)



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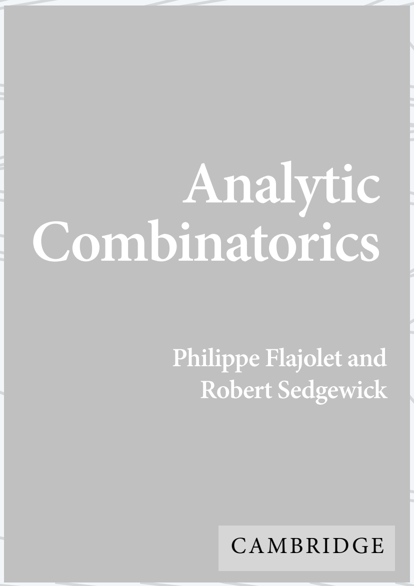
4. Complex Analysis, Rational and Meromorphic functions

- Roadmap
- Complex functions
- **Rational functions**
- Analytic functions and complex integration
- Meromorphic functions



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4. Complex Analysis, Rational and Meromorphic functions

- Roadmap
- Complex functions
- Rational functions
- **Analytic functions and complex integration**
- Meromorphic functions

Analytic functions

Definition. A function $f(z)$ defined in Ω is *analytic* at a point z_0 in Ω iff for z in an open disc in Ω centered at z_0 it is representable by a power-series expansion $f(z) = \sum_{N\geq 0} c_N (z - z_0)^N$

Definition. A *singularity* is a point where a function ceases to be analytic.

Example: $$\frac{1}{1-z} = \sum_{N\geq 0} z^N$$ ← analytic at 0

$$\frac{1}{1-z} = \frac{1}{1-z_0-(z-z_0)} = \frac{1}{1-z_0}\frac{1}{1-\frac{z-z_0}{1-z_0}}$$

$$= \sum_{N\geq 0}\Big(\frac{1}{1-z_0}\Big)^{N+1}(z-z_0)^N$$ ← analytic everywhere but $z = 1$

Analytic functions

Definition. A function $f(z)$ defined in Ω is *analytic* at a point z_0 in Ω iff for z in an open disc in Ω centered at z_0 it is representable by a power-series expansion $f(z) = \sum_{N \geq 0} c_N (z - z_0)^N$

function	region of meromorphicity
$1 + z + z^2$	everywhere
$\frac{1}{z}$	everywhere but $z = 0$
$D(z) = \frac{e^{-z}}{1-z}$	everywhere but $z = 1$
$\frac{1}{1+z^2}$	everywhere but $z = \pm i$
$I(z) = e^{z+z^2/2}$	everywhere
$S_r(z) = \frac{z^r}{(1-z)(1-2z)\ldots(1-rz)}$	everywhere but $z = 1, 1/2, 1/3, \ldots$
$G(z) = \frac{1+\sqrt{1-4z}}{2}$	everywhere but $z = 1/4$
$R(z) = \frac{1}{2-e^z}$	everywhere but $z = \ln 2 \pm 2\pi k i$
$C(z) = \frac{1}{1-z} \ln \frac{1}{1-z}$	everywhere but $z = 1$

Aside: computing with complex functions

is an easy exercise in object-oriented programming.

```
public class Complex
{
    private final double re;   // real part
    private final double im;   // imaginary part

    public Complex(double real, double imag)
    {
        re = real;
        im = imag;
    }

    public Complex plus(Complex b)
    {
        Complex a = this;
        double real = a.re + b.re;
        double imag = a.im + b.im;
        return new Complex(real, imag);
    }

    public Complex times(Complex b)
    {
        Complex a = this;
        double real = a.re * b.re - a.im * b.im;
        double imag = a.re * b.im + a.im * b.re;
        return new Complex(real, imag);
    }

    ...
}
```

```
public interface ComplexFunction
{
   public Complex eval(Complex z);
}
```

```
public class Example implements ComplexFunction
{
    public Complex eval(Complex z)
    {  // {1 \over 1+z^3}
        Complex one = new Complex(1, 0);
        Complex d = one.plus(z.times(z.times(z)));
        return d.reciprocal();
    }
}
```

Design choice: complex numbers are *immutable*

- create a new object for every computed value
- object value never changes

[Same approach as for Java strings.]

Aside (continued): plotting complex functions

is also an easy (and instructive!) programming exercise.

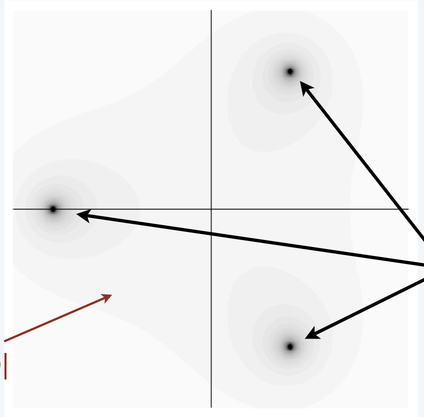
```
public class Plot2Dez
{
  public static void show(ComplexFunction f, int sz)
  {
   StdDraw.setCanvasSize(sz, sz);
   StdDraw.setXscale(0, sz);
   StdDraw.setYscale(0, sz);
   double scale = 2.5;
   for (int i = 0; i < sz; i++)
      for (int j = 0; j < sz; j++)
      {
         double x = ((1.0*i)/sz - .5)*scale;
         double y = ((1.0*j)/sz - .5)*scale;
         Complex z = new Complex(x, y);
         double val = f.eval(z).abs()*10;
         int t;
         if  (val <   0) t = 255;
         else if (val > 255) t = 0;
         else t = 255 - (int) val;
         Color c = new Color(t, t, t);
         StdDraw.setPenColor(c);
         StdDraw.pixel(i, j);
      }
   Color c = new Color(0, 0, 0);
   StdDraw.setPenColor(c);
   StdDraw.line(sz/2, 0, sz/2, sz);
   StdDraw.line(0, sz/2, sz, sz/2);
   StdDraw.show();
  }
}
```

arbitrary factor to emphasize growth

```
public class Example implements ComplexFunction
{
    public Complex eval(Complex z)
    {  // {1 \over 1+z^3}
        Complex one = new Complex(1, 0);
        Complex d = one.plus(z.times(z.times(z)));
        return d.reciprocal();
    }
   public static void main(String[] args)
   {  Plot2D.show(new Example(), 512);  }
}
```

our convention:
plots are in the 2.5 by 2.5 square centered at the origin

$$\frac{1}{1+z^3}$$

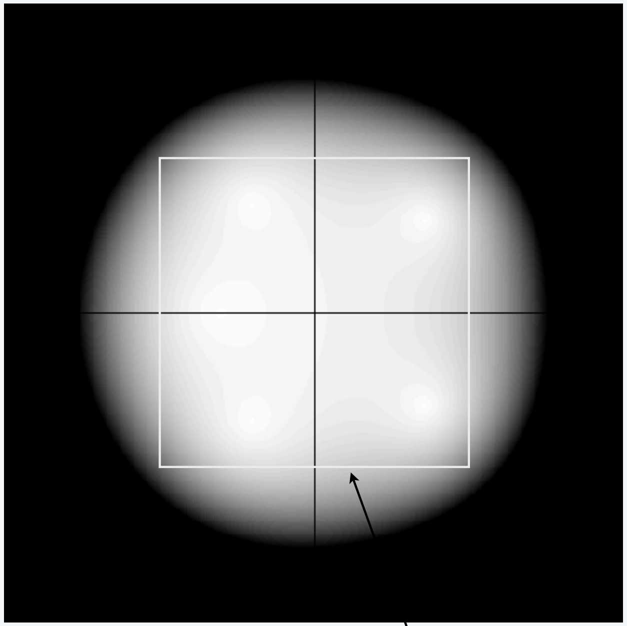


singularities (where $|f| \to \infty$)

darkness of pixel at (x, y) is proportional to $|f(x+iy)|$

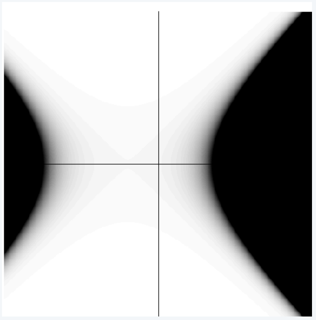
Entire functions (analytic everywhere)

$1 + z + z^5$



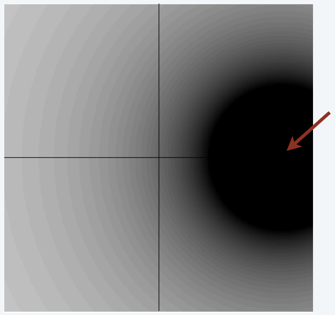
our convention:
highlight the 2.5 by 2.5 square
centered at the origin
when plotting a bigger square

$e^{z+z^2/2}$

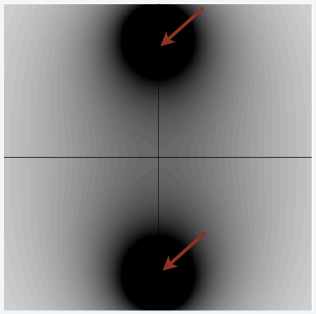


Plots of various rational functions

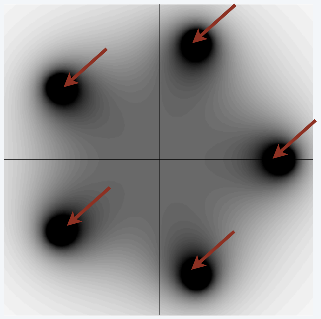
$$\frac{1}{1-z}$$



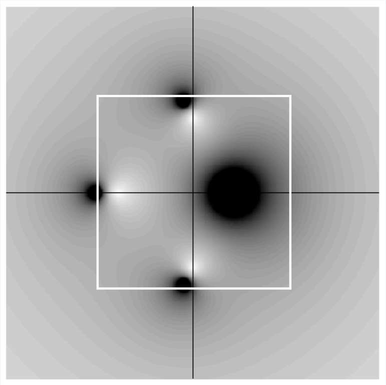
$$\frac{1}{1+z^2}$$



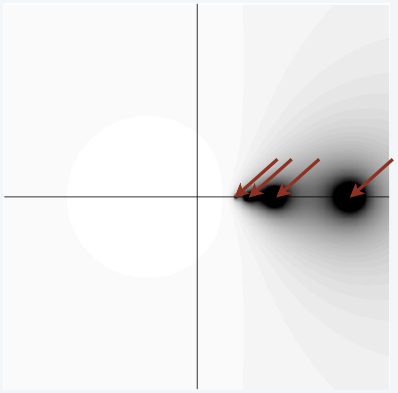
$$\frac{1}{1-z^5}$$



$$\frac{1+z+z^2+z^3}{1-z-z^2-z^3-z^4}$$

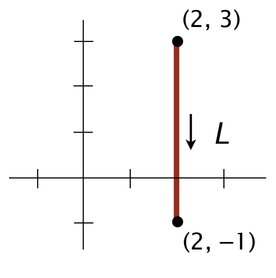


$$\frac{z^4}{(1-z)(1-2z)(1-3z)(1-4z)}$$



Complex integration

Starting point: Change variables to convert to real integrals.



$$\int_L z\,dz = \int_3^{-1} (2+iy)\,i\,dy$$

$z = x + iy \quad dz = i\,dy$

$$= 2i - \frac{y^2}{2}\Big|_3^{-1} = 2i + 4$$

Augustin-Louis Cauchy
1789–1857

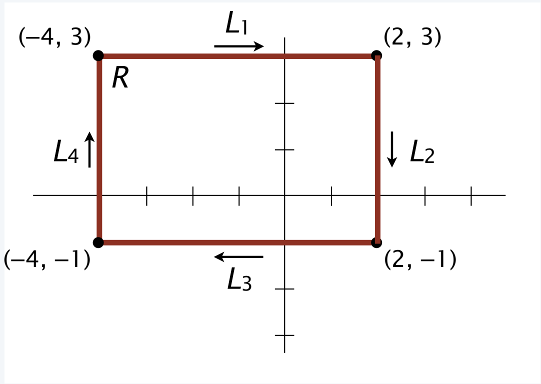
Amazing facts:

- *The integral of an analytic function around a loop is 0.*
- *The coefficients of an analytic function can be extracted via complex integration*

Analytic combinatorics context: *Immediately* gives exponential growth for meromorphic GFs

Integration examples

Ex 1. Integrate $f(z) = z$ on a rectangle



$$\int_{L_1} zdz = \int_{-4}^{2} xdx + 3i = \frac{x^2}{2}\Big|_{-4}^{2} + 3i = -6 + 3i$$

$$\int_{L_2} zdz = \int_{3}^{-1} (2 + iy)idy = 2i - \frac{y^2}{2}\Big|_{3}^{-1} = 2i + 4$$

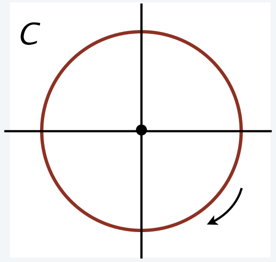
$z = x + iy \quad dz = idy$

$$\int_{L_3} zdz = \int_{2}^{-4} xdx - i = \frac{x^2}{2}\Big|_{2}^{-4} - i = 6 - i$$

$$\int_{L_4} zdz = \int_{-1}^{3} (-4 + iy)idy = -4i - \frac{y^2}{2}\Big|_{-1}^{3} = -4i - 4$$

$$\int_R zdz = \int_{L_1+L_2+L_3+L_4} zdz = -6 + 3i + 2i + 4 + 6 - i - 4i - 4 = 0 \quad (!)$$

$z = re^{i\theta} \quad dz = ire^{i\theta}d\theta$



Ex 2. Integrate $f(z) = z$ on a circle centered at 0

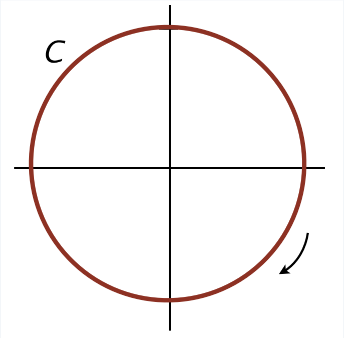
$$\int_C zdz = ir^2 \int_0^{2\pi} e^{2i\theta}d\theta = \frac{e^{2i\theta}}{2i}\Big|_0^{2\pi} = \frac{1}{2i}(1 - 1) = 0$$

Ex 3. Integrate $f(z) = 1/z$ on a circle centered at 0

$$\int_C \frac{dz}{z} = i\int_0^{2\pi} d\theta = 2\pi i$$

Integration examples

Ex 4. Integrate $f(z) = z^M$ on a circle centered at 0



$$\int_C z^M dz = ir^{M+1} \int_0^{2\pi} e^{i(M+1)\theta} d\theta$$

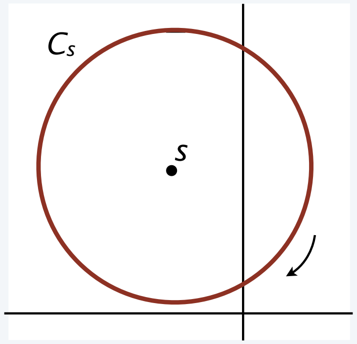
$$z = re^{i\theta} \quad dz = ire^{i\theta} d\theta$$

$$= \begin{cases} 2\pi i & \text{if} \quad M = -1 \\ 0 & \text{if} \quad M \neq -1 \end{cases}$$

$$\int_0^{2\pi} d\theta = 2\pi$$

$$\int_0^{2\pi} e^{(M+1)i\theta} d\theta = \frac{e^{(M+1)i\theta}}{(M+1)i}\Big|_0^{2\pi} = \frac{1}{(M+1)i}(1-1) = 0$$

Ex 5. Integrate $f(z) = (z-s)^M$ on a circle centered at s



$$\int_{C_s} (z-s)^M dz = ir^{M+1} \int_0^{2\pi} e^{i(M+1)\theta} d\theta$$

$$z - s = re^{i\theta} \quad dz = ire^{i\theta} d\theta$$

$$= \begin{cases} 2\pi i & \text{if} \quad M = -1 \\ 0 & \text{if} \quad M \neq -1 \end{cases}$$

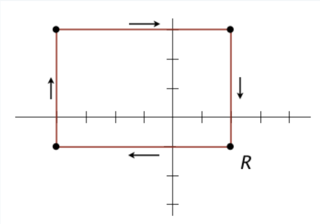
Null integral property

Theorem. (Null integral property).

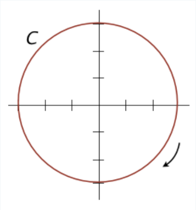
If $f(z)$ is analytic in Ω then $\int_\lambda f(z)dz = 0$ for *any* closed loop λ in Ω.

For purposes of this lecture: Axiom 2.

Ex. $f(z) = z$



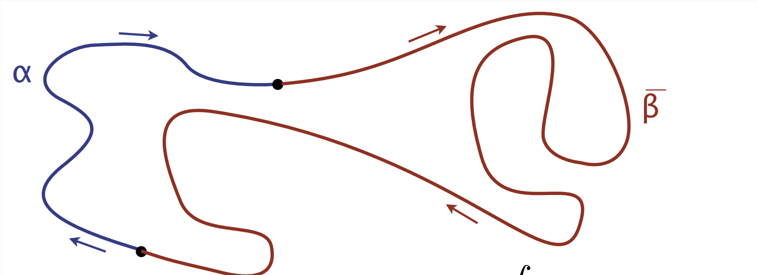
$$\int_R z\,dz = 0$$



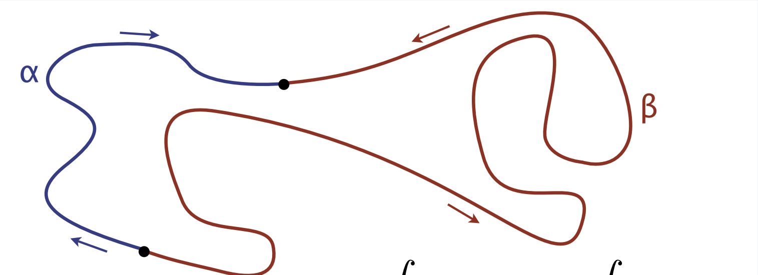
$$\int_C z\,dz = 0$$

Equivalent fact: $\int_\alpha f(z)dz = \int_\beta f(z)dz$ for *any* homotopic paths α and β in Ω.

Homotopic: Paths that can be continuously deformed into one another.

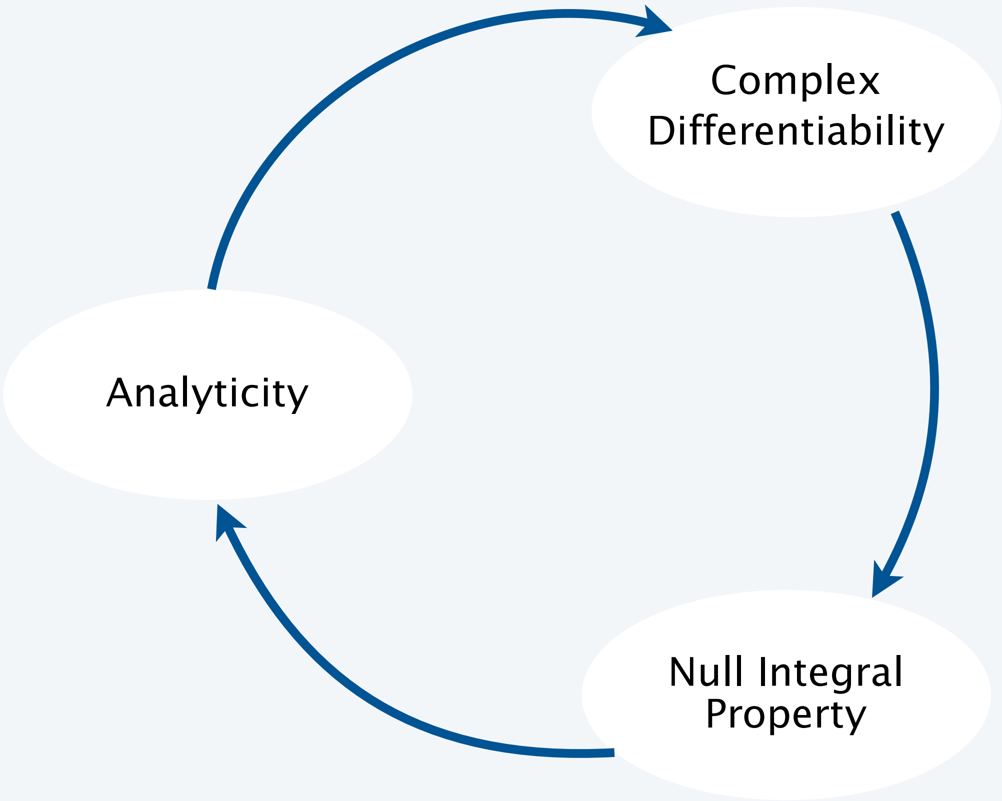


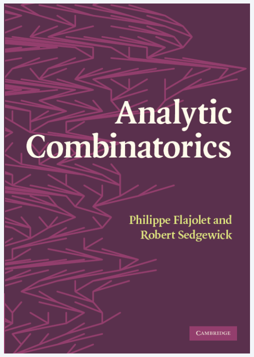
$$\int_{\alpha+\overline{\beta}} f(z)dz = 0$$



$$\int_\alpha f(z)dz = \int_\beta f(z)dz$$

Deep theorems of complex analysis





Appendix C
pp. 741-743

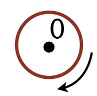
Cauchy's coefficient formula

Theorem. If $f(z)$ is analytic and λ is a closed +loop in a region Ω that contains 0, then

$$f_n \equiv [z^n]f(z) = \frac{1}{2\pi i}\int_\lambda f(z)\frac{dz}{z^{n+1}}$$

Proof.

- Expand f: $f(z) = f_0 + f_1 z + f_2 z^2 + f_3 z^3 + \ldots$
- Deform λ to a circle centered at 0



- Integrate: $\int_{C_s} f(z)\frac{dz}{z^{n+1}} = \int_{C_s}\left(\frac{f_0}{z^{n+1}} + \ldots + \frac{f_n}{z} + f_{n+1} + f_{n+2}z + \ldots\right)dz$

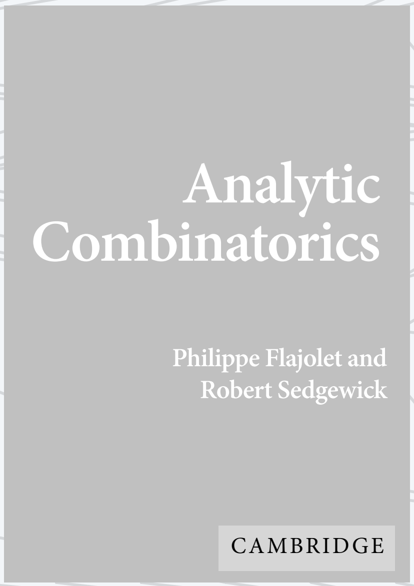
$$= 2\pi\, i\, f_n$$

← See integration example 4

AC context: provides transfer theorems for broader class of complex functions: *meromorphic* functions (next).

ANALYTIC COMBINATORICS

PART TWO



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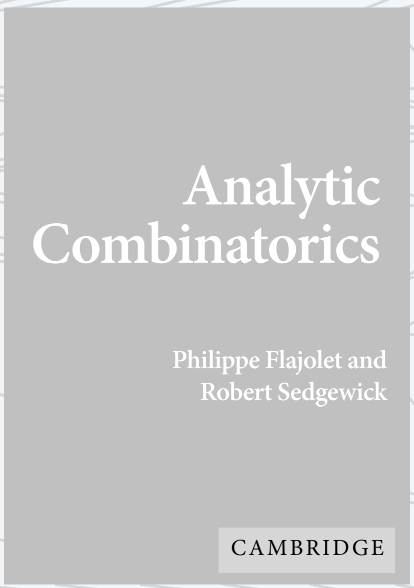
II.4d.CARM.Analytic



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4. Complex Analysis, Rational and Meromorphic functions

- Roadmap
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Meromorphic functions

are complex functions that can be expressed as the ratio of two *analytic functions*.

Note: All rational functions are meromorphic.

$$D(z) = \frac{e^{-z}}{1-z} \qquad G(z) = \frac{1+\sqrt{1-4z}}{2} \qquad R(z) = \frac{1}{2-e^z} \qquad B_P(z) = \frac{1+z+z^2+\ldots+z^{P-1}}{1-z-z^2-\ldots-z^P}$$

$$S_r(z) = \frac{z^r}{(1-z)(1-2z)\ldots(1-rz)} \qquad C(z) = \frac{1}{1-z}\ln\frac{1}{1-z} \qquad I(z) = e^{z+z^2/2}$$

Approach:

- Use *contour integration* to expand into terms for which coefficient extraction is easy.
- Focus on the largest term to approximate.

[Same approach as for rationals, resulting in a more general transfer theorem.]

Meromorphic functions

Definition. A function $h(z)$ defined in Ω is *meromorphic* at z_0 in Ω iff for z in a neighborhood of z_0 with $z \neq z_0$ it can be represented as $f(z)/g(z)$, where $f(z)$ and $g(z)$ are analytic at z_0.

Useful facts:

- A function $h(z)$ that is meromorphic at z_0 admits an expansion of the form

$$h(z) = \frac{h_{-M}}{(z-z_0)^M} + \ldots + \frac{h_{-2}}{(z-z_0)^2} + \frac{h_{-1}}{(z-z_0)} + h_0 + h_1(z-z_0) + h_2(z-z_0)^2 + \ldots$$

  and is said to have a pole of order M at z_0.

Proof sketch: If z_0 is a zero of $g(z)$ then $g(z) = (z - z_0)^M G(z)$.
Expand the analytic function $f(z)/G(z)$ at z_0.

- The coefficient h_{-1} is called the residue of $h(z)$ at z_0, written $\operatorname*{Res}_{z=z_0} h(z)$.
- If $h(z)$ has a pole of order M at z_0, the function $(z - z_0)^M h(z)$ is analytic at z_0.

A function is meromorphic in Ω iff it is analytic in Ω *except for a set of isolated singularities*, its *poles*.

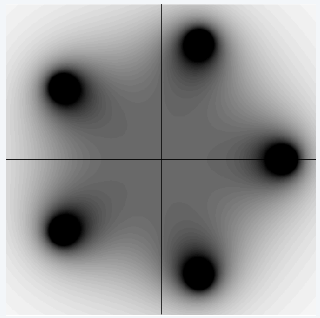
Meromorphic functions

Definition. A function $h(z)$ defined in Ω is *meromorphic* at z_0 in Ω iff for z in a neighborhood of z_0 with $z \neq z_0$ it can be represented as $f(z)/g(z)$, where $f(z)$ and $g(z)$ are analytic at z_0.

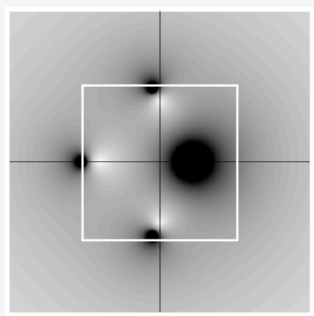
function	region of meromorphicity
$1 + z + z^2$	everywhere
$\frac{1}{z}$	everywhere but $z = 0$
$D(z) = \frac{e^{-z}}{1 - z}$	everywhere but $z = 1$
$\frac{1}{1 + z^2}$	everywhere but $z = \pm i$
$S_r(z) = \frac{z^r}{(1 - z)(1 - 2z) \ldots (1 - rz)}$	everywhere but $z = 1, 1/2, 1/3, \ldots$
$R(z) = \frac{1}{2 - e^z}$	everywhere but $z = \ln 2 \pm 2\pi k i$

Plots of various meromorphic functions

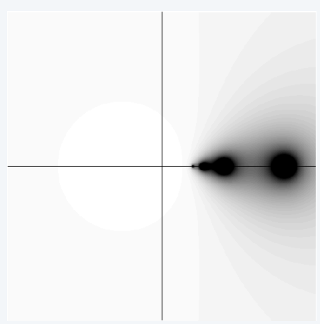
$$\frac{1}{1-z^5}$$



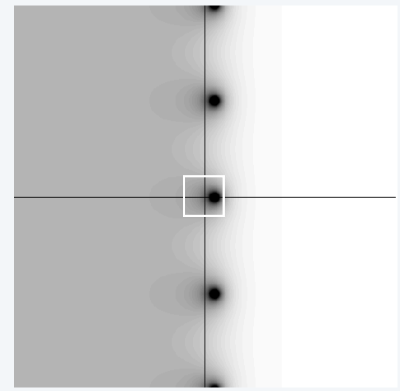
$$\frac{1+z+z^2+z^3}{1-z-z^2-z^3-z^4}$$



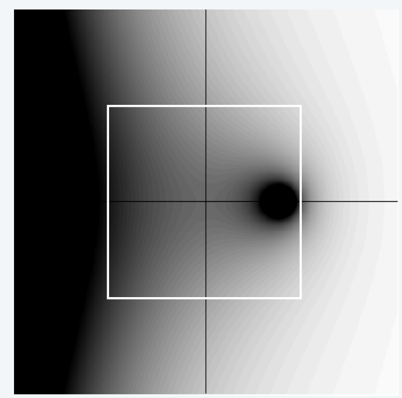
$$\frac{z^4}{(1-z)(1-2z)(1-3z)(1-4z)}$$



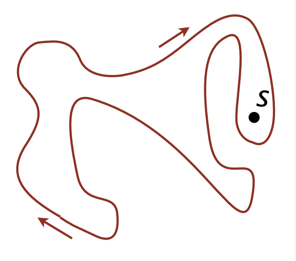
$$\frac{1}{2-e^z}$$



$$\frac{e^{-z}}{1-z}$$

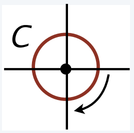


Integrating around a pole



Lemma. If $h(z)$ is meromorphic and λ is a closed +loop with a single pole s of h inside, then $\displaystyle\int_\lambda h(z)dz = 2\pi i \operatorname*{Res}_{z=s} h(z)$

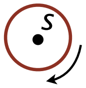
Ex. $f(z) = 1/z$, pole at 0 with residue 1.



$$\int_C \frac{1}{z}\, dz = 2\pi i$$

Proof.

- Expand h: $\displaystyle h(z) = \frac{h_{-M}}{(z-s)^M} + \ldots + \frac{h_{-1}}{(z-s)} + h_0 + h_1(z-s) + h_2(z-s)^2 + \ldots$
- Deform λ to a circle centered at s that contains no other poles



- Integrate: $\displaystyle\int_{C_s} h(z)dz = \int_{C_s} \Big(\frac{h_{-M}}{(z-s)^M} + \ldots + \frac{h_{-1}}{(z-s)} + h_0 + h_1(z-s) + h_2(z-s)^2 + \ldots\Big) dz$

$$= 2\pi\, i\, h_{-1}$$

← See integration example 5

Significance: Connects *local* properties of a function (residue at a point) to *global* properties elsewhere (integral along a distant curve).

Residue theorem

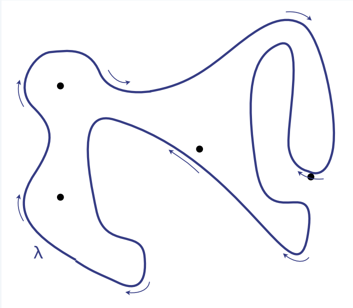
Theorem. If $h(z)$ is meromorphic and λ is a closed +loop in Ω, then

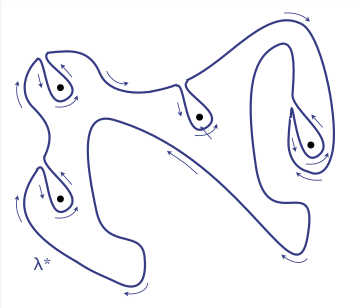
$$\frac{1}{2\pi i}\int_{\lambda} h(z)dz = \sum_{s\in S} \operatorname*{Res}_{z=s} h(z)$$

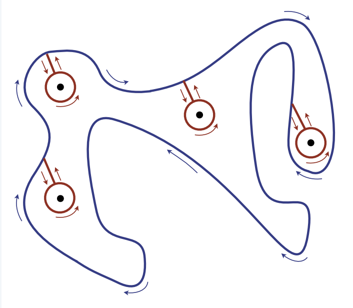
where S is the set of poles of $h(z)$ inside Ω

Proof (sketch).

- Consider small circles C_s centered at each pole.
- Define a path λ* that follows λ but travels in, around, and out each C_s.
- Poles are all outside λ* so integral around λ* is 0.
- Paths in and out cancel, so $\displaystyle\int_{\lambda^*} h(z)dz = \int_{\lambda} h(z)dz - \sum_{s\in S}\int_{C_s} h(z)dz = 0$
- By the single-pole lemma $\displaystyle\int_{C_s} h(z)dz = 2\pi i \operatorname*{Res}_{z=s} h(z)$





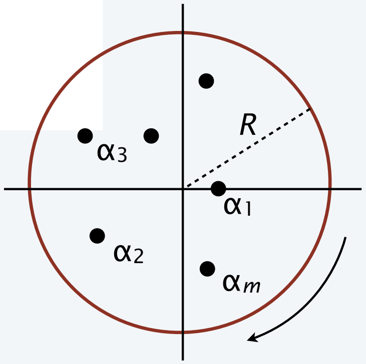


Extracting coefficients from meromorphic GFs

Theorem. Suppose that $h(z)$ is meromorphic in the closed disc $|z| \leq R$; analytic at $z = 0$ and all points $|z| = R$; and that $\alpha_1, \ldots \alpha_m$ are the poles of $h(z)$ in R. Then

$$h_N \equiv [z^N]h(z) = \frac{p_1(N)}{\alpha_1^N} + \frac{p_2(N)}{\alpha_2^N} + \ldots + \frac{p_m(N)}{\alpha_m^N} + O\left(\frac{1}{R^N}\right)$$

where $p_1, \ldots, p_m$ are polynomials with degree $\alpha_1-1, \ldots, \alpha_m-1$, respectively.



Proof sketch:

- Consider the integral $\quad I_{RN} = \frac{1}{2\pi i}\int_{|z|=R} h(z)\frac{dz}{z^{N+1}}$
- By the residue theorem $\quad I_{RN} = \sum_{1\leq i\leq m} \operatorname*{Res}_{z=\alpha_i} \frac{h(z)}{z^{N+1}} = \frac{p_1(N)}{\alpha_1^N} + \ldots + \frac{p_m(N)}{\alpha_m^N}$
- By direct bound $\quad I_{RN} < \frac{A}{R^N}$ where $|h(z)| < A$ for $|z| = R$

Ex. If α_i is order 1

$$h(z) \sim \frac{c}{(z-\alpha_i)} \text{ as } z \to \alpha_i$$

$$\operatorname*{Res}_{z=\alpha_i} \frac{h(z)}{z^{N+1}} = \operatorname*{Res}_{z=\alpha_i} \frac{c}{z^{N+1}(z-\alpha_i)} = \frac{c}{\alpha_i^{N+1}}$$

Constant. May depend on R, but *not* N.

Complex roots

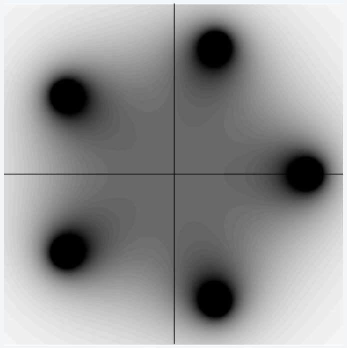
Q. Do complex roots introduce complications in deriving asymptotic estimates of coefficients?

A. YES: *all* poles closest to the origin contribute to the leading term.

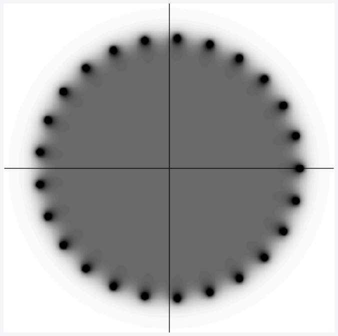
Prime example: *N*th roots of unity $r_{kN} = \exp(\frac{2\pi ik}{N}) = \cos(\frac{2\pi ik}{N}) + i\sin(\frac{2\pi ik}{N})$ for $0 \le k < N$

all are distance 1 from origin with $(r_{kN})^N = 1$

$\frac{1}{1-z^5}$

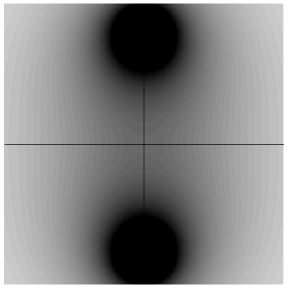


$\frac{1}{1-z^{25}}$



Rational GF example earlier in this lecture.

$\frac{1}{1+z^2}$



$[z^N]\frac{1}{1+z^2} = 1, 0, -1, 0, 1, 0, -1, \ldots$

Complex roots

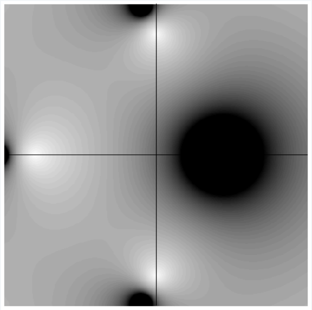
Q. Do complex roots introduce complications in deriving asymptotic estimates of coefficients?

A. NO, for combinatorial GFs, if only one root is closest to the origin.

Pringsheim's Theorem. If $h(z)$ can be represented as a series expansion in powers of z *with non-negative coefficients* and radius of convergence R, then the point $z = R$ is a singularity of $h(z)$.

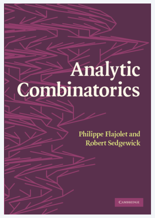
smallest positive real root

$$\frac{1+z+z^2+z^3}{1-z-z^2-z^3-z^4}$$



Implication: *Only the smallest positive real root matters* if no others have the same magnitude.

If some *do* have the same magnitude, complicated periodicities can be present. See "Daffodil Lemma" on page 266.



AC transfer theorem for meromorphic GFs (leading term)

Theorem. Suppose that $h(z) = f(z)/g(z)$ is meromorphic in $|z| \leq R$ and analytic both at $z = 0$ and at all points $|z| = R$. If α is a unique closest pole to the origin of $h(z)$ in R, then α is real and $[z^N]\dfrac{f(z)}{g(z)} \sim c\beta^N N^{M-1}$ where M is the order of α, $c = (-1)^M \dfrac{Mf(\alpha)}{\alpha^M g^{(M)}(\alpha)}$ and $\beta = 1/\alpha$.

Proof sketch for $M = 1$:

- Series expansion (valid near α): $h(z) = \dfrac{h_{-1}}{\alpha - z} + h_0 + h_1(\alpha - z) + h_2(\alpha - z)^2 + \ldots$ ← elementary from Pringsheim's and coefficient extraction theorems
- One way to calculate constant: $h_{-1} = \lim_{z \to \alpha}(\alpha - z)h(z)$
- Approximation at α: $h(z) \sim \dfrac{h_{-1}}{\alpha - z} = \dfrac{1}{\alpha}\dfrac{h_{-1}}{1 - z/\alpha} = \dfrac{h_{-1}}{\alpha}\sum_{N \geq 0}\dfrac{z^N}{\alpha^N}$

See next slide for calculation of c and $M > 1$.

Notes:

- Error is *exponentially small* (and next term may involve periodicities due to complex roots).
- Result is the same as for rational functions.

Computing coefficients for a meromorphic function $h(z) = f(z)/g(z)$ at a pole α

If α is of order 1 then $h_N \equiv [z^N]h(z) \sim \dfrac{h_{-1}}{\alpha^{N+1}}$ where $h_{-1} = \lim_{z\to\alpha}(\alpha - z)h(z)$

To calculate h_{-1}: $\lim_{z\to\alpha}(\alpha - z)h(z) = \lim_{z\to\alpha}\dfrac{(\alpha - z)f(z)}{g(z)} = \lim_{z\to\alpha}\dfrac{(\alpha - z)f'(z) - f(z)}{g'(z)} = -\dfrac{f(\alpha)}{g'(\alpha)}$

If α is of order 2 then $h_N \equiv [z^N]h(z) \sim h_{-2}\dfrac{N}{\alpha^{N+2}}$ where $h_{-2} = \lim_{z\to\alpha}(\alpha - z)^2 h(z)$

Series expansion (valid near α): $h(z) = \dfrac{h_{-2}}{(\alpha - z)^2} + \dfrac{h_{-1}}{\alpha - z} + h_0 + h_1(\alpha - z) + h_2(\alpha - z)^2 + \ldots$

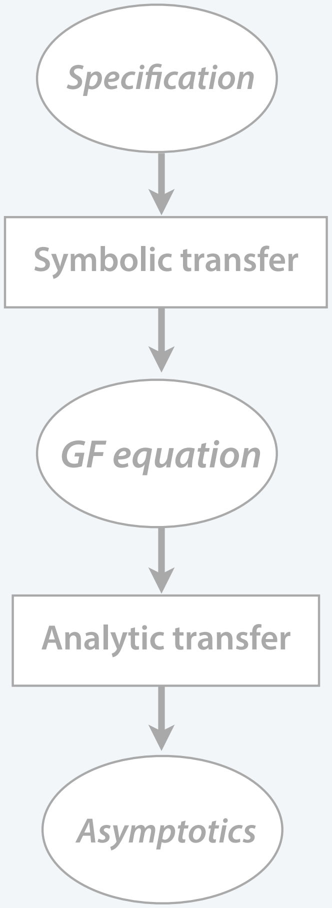
Approximation at α: $h(z) \sim \dfrac{h_{-2}}{(\alpha - z)^2} = \dfrac{1}{\alpha^2}\dfrac{h_{-2}}{(1 - z/\alpha)^2} = \dfrac{h_{-2}}{\alpha^2}\sum_{N\geq 0}\dfrac{(N+1)z^N}{\alpha^N}$

To calculate h_{-2}: $\lim_{z\to\alpha}(\alpha - z)^2 h(z) = \lim_{z\to\alpha}\dfrac{(\alpha - z)^2 f(z)}{g(z)} = \lim_{z\to\alpha}\dfrac{(\alpha - z)^2 f'(z) - 2(\alpha - z)f(z)}{g'(z)}$

$$= \lim_{z\to\alpha}\frac{(\alpha - z)^2 f''(z) - 4(\alpha - z)f'(z) + 2f(z)}{g''(z)} = \frac{2f(\alpha)}{g''(\alpha)}$$

If α is of order M then $h_N \equiv [z^N]h(z) \sim (-1)^M \dfrac{Mf(\alpha)}{g^{(M)}(\alpha)\alpha^M} N^{M-1}\left(\dfrac{1}{\alpha}\right)^N$

Bottom line



Analytic transfer for meromorphic GFs: $f(z)/g(z) \sim c\,\beta^N$

- Compute the dominant pole α (smallest real with $g(z) = 0$).
- (Check that no others have the same magnitude.)
- Compute the residue $h_{-1} = -f(\alpha)/g'(\alpha)$.
- Constant c is h_{-1}/α.
- Exponential growth factor β is $1/\alpha$

Not order 1 if $g'(\alpha) = 0$.
Adjust to (slightly) more complicated order M case.

AC transfer for meromorphic GFs

Analytic transfer for meromorphic GFs: $f(z)/g(z) \sim c\,\beta^N$

- Compute the dominant pole α (smallest real with $g(z) = 0$).
- (Check that no others have the same magnitude.)
- Compute the residue $h_{-1} = -f(\alpha)/g'(\alpha)$.
- Constant c is h_{-1}/α.
- Exponential growth factor β is $1/\alpha$

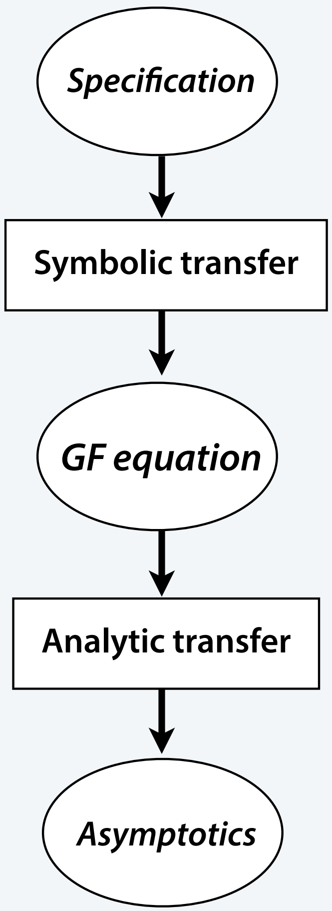
Examples.

$h(z) = f(z)/g(z)$	α	h_{-1}	$[z^N]h(z)$
$\dfrac{z}{1-z-z^2}$	$\hat\phi = \dfrac{1}{\phi}$	$\dfrac{\hat\phi}{(1+2\hat\phi)} = \dfrac{\hat\phi}{\sqrt5}$	$\sim \dfrac{1}{\sqrt5}\phi^N$
$\dfrac{e^{-z}}{1-z}$	1	$\dfrac{1}{e}$	$\dfrac{1}{e}$
$\dfrac{e^{-z-z^2/2-z^3/3}}{1-z}$	1	$\dfrac{1}{e^{H_3}}$	$\dfrac{1}{e^{H_3}}$

$$\hat\phi = \frac{\sqrt5 - 1}{2}$$

$$\phi = \frac{\sqrt5 + 1}{2}$$

AC example with meromorphic GFs: Generalized derangements



D_M, the class of all permutations with no cycles of length ≤ M

see Lecture 2

D_M = SET(CYC$_{>M}$(***Z***))

$$D_M(z) = \frac{e^{-z-\frac{z^2}{2}-\frac{z^3}{3}-\dots\frac{z^M}{M}}}{1-z}$$

$$[z^N]D_M(z) \sim e^{-H_M}$$

Many, many more examples to follow (next lecture)

General form of coefficients of combinatorial GFs (revisited)

$$[z^N]F(z) = A^N\theta(N)$$

exponential growth factor: A^N

subexponential factor: $\theta(N)$

First principle of coefficient asymptotics

The *location* of a function's singularities dictates the *exponential growth* of its coefficients.

Second principle of coefficient asymptotics

The *nature* of a function's singularities dictates the *subexponential factor* of the growth.

When F(z) is a meromorphic function $f(z)/g(z)$

- If the smallest real root of $g(z)$ is α then the exponential growth factor is $1/\alpha$.
- If α is a pole of order M, then the subexponential factor is cN^{M-1}.

Parting thoughts

"Despite all appearances, generating functions belong to algebra, not analysis"

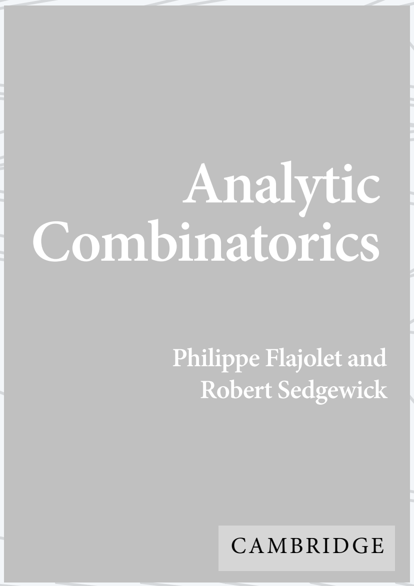
— John Riordan, 1958

"Combinatorialists use recurrences, generating functions, and such transformations as the Vandermonde convolution; Others, to my horror, use contour integrals, differential equations, and other resources of mathematical analysis"

— John Riordan, 1968

$$[z^N]\frac{e^{-z}}{1-z} = [z^N]\sum_{k_1\geq 0} z^{k_1} \sum_{k_2\geq 0}\frac{(-z)^{k_2}}{k_2!} = \sum_{0\leq k\leq N}\frac{(-1)^k}{k!} \sim \frac{1}{e}$$

$$[z^N]\frac{e^{-z-z^2/2-z^3/3}}{1-z} = [z^N]\sum_{k_1\geq 0} z^{k_1} \sum_{k_2\geq 0}\frac{(-z)^{k_2}}{k_2!}\sum_{k_3\geq 0}\frac{(-z)^{k_3}}{2^{k_3}k_3!}\sum_{k_4\geq 0}\frac{(-z)^{k_4}}{3^{k_4}k_4!} = \ldots$$



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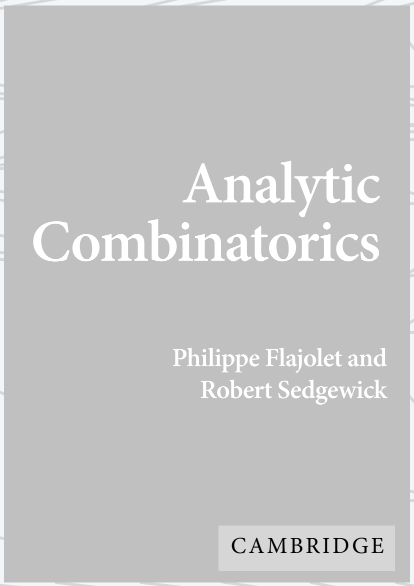
4. Complex Analysis, Rational and Meromorphic functions

- Roadmap
- Complex functions
- Rational functions
- Analytic functions and complex integration
- **Meromorphic functions**



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4. Complex Analysis, Rational and Meromorphic functions

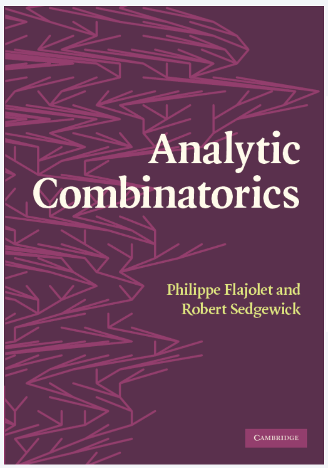
- Roadmap
- Complex functions
- Rational functions
- Analytic functions and complex integration
- Meromorphic functions
- **Exercises**

Note IV.28

Supernecklaces

Warmup: A "supernecklace" of the 3rd type is a labelled cycle of cycles.

Draw all the supernecklaces of the 3rd type of size N for $N = 1, 2, 3,$ and 4.



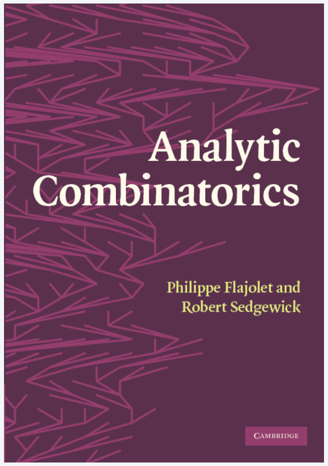
▷ **IV.28.** *Some "supernecklaces"*. One estimates

$$[z^n] \log\left(\frac{1}{1 - \log \frac{1}{1-z}}\right) \sim \frac{1}{n}(1 - e^{-1})^{-n},$$

where the EGF enumerates labelled cycles of cycles (supernecklaces, p. 125). [Hint: Take derivatives.] ◁

Assignments

1. Read pages 223-288 (*Complex Analysis, Rational, and Meromorphic Functions*) in text.
   Usual caveat: Try to get a feeling for what's there, not understand every detail.



2. Write up solution to Note IV.28.

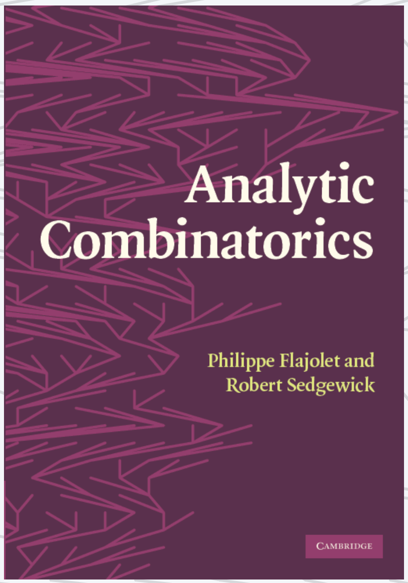
3. Programming exercises.

Program IV.1. Compute the percentage of permutations having no singelton or doubleton cycles and compare with the AC asymptotic estimate, for N = 10 and N = 20. .

Program IV.2. Plot the derivative of the supernecklace GF (see Note IV.28) in the style of the plots in this lecture (see booksite for Java code).

ANALYTIC COMBINATORICS

PART TWO



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4. Complex Analysis, Rational and Meromorphic Asymptotics